The automorphism group of the free group of rank two is a CAT(0) group

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February 25, 2009

Abstract

We prove that the automorphism group of the braid group on four strands acts faithfully and geometrically on a CAT(0) 2-complex. This implies that the automorphism group of the free group of rank two acts faithfully and geometrically on a CAT(0) 2-complex, in contrast to the situation for rank three and above.

1 Introduction

A CAT(0) metric space is a proper complete geodesic metric space in which each geodesic triangle with side lengths $a$, $b$ and $c$ is “at least as thin” as the Euclidean triangle with side lengths $a$, $b$ and $c$ (see [5] for details). We say that a finitely generated group $G$ is a $CAT(0)$ group if $G$ may be realized as a cocompact and properly discontinuous subgroup of the isometry group of a CAT(0) metric space $X$. Equivalently, $G$ is a CAT(0) group if there exists a CAT(0) metric space $X$ and a faithful geometric action of $G$ on $X$. It is perhaps not standard to require that the group action be faithful, a point which we address in Remark 1 below.

For each integer $n \geq 2$, we write $F_n$ for the free group of rank $n$ and $B_n$ for the braid group on $n$ strands.

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In [3], T. Brady exhibited a subgroup $H \leq \text{Aut}(F_2)$ of index 24 which acts faithfully and geometrically on CAT(0) 2-complex. In subsequent work [4], the same author showed that $B_4$ acts faithfully and geometrically on a CAT(0) 3-complex. It follows that $\text{Inn}(B_4)$ acts faithfully and geometrically on a CAT(0) 2-complex $X_0$ (this fact is explained explicitly by Crisp and Paoluzzi in [8]). Now, $\text{Inn}(B_n)$ has index two in $\text{Aut}(B_n)$ [10], and $\text{Aut}(F_2)$ is isomorphic to $\text{Aut}(B_4)$ [16, 10], thus the result in the title of this paper is proved if we exhibit an extra isometry of $X_0$ which extends the faithful geometric action of $\text{Inn}(B_4)$ to a faithful geometric action of $\text{Aut}(B_4)$. We do this in §2 below.

In the language of [14], $X_0$ is a systolic simplicial complex. By [14, Theorem 13.1], a group which acts simplicially, properly discontinuously and cocompactly on such a space is biautomatic. Since the action of $\text{Aut}(F_2)$ provided here is of this type, it follows that $\text{Aut}(F_2)$ is biautomatic.

Our results reinforce the striking contrast between those properties enjoyed by $\text{Aut}(F_2)$ and those enjoyed by the automorphism groups of finitely generated free groups of higher ranks. We can now say that $\text{Aut}(F_2)$ is a CAT(0) group, a biautomatic group and it has a faithful linear representation [9, 16]; while $\text{Aut}(F_n)$ is not a CAT(0) group [12], nor a biautomatic group [6] and it does not have a faithful linear representation [11] whenever $n \geq 3$.

We regard the CAT(0) 2-complex $X_0$ as a geometric companion to Auter Space (of rank two) [13], a topological construction equipped with a group action by $\text{Aut}(F_2)$.

Let $W_3$ denote the universal Coxeter group of rank 3—that is, $W_3$ is the free product of 3 copies of the group of order two. Since $\text{Aut}(F_3)$ is isomorphic to $\text{Aut}(W_3)$ (see Remark 2 below), we also learn that $\text{Aut}(W_3)$ is a CAT(0) group.

**Remark 1.** As pointed out in the opening paragraph, our definition of a CAT(0) group is perhaps not standard because of the requirement that the group action be faithful. We note that such a requirement is redundant when giving an analogous definition of a word hyperbolic group. This follows from the fact that word hyperbolicity is an invariant of the quasi-isometry class of a group. In contrast, the CAT(0) property is not an invariant of the quasi-isometry class of a group. Examples are known of two quasi-isometric groups, one of which is CAT(0), and the other of which is not. Examples of this type may be constructed using the fundamental groups of graph manifolds [15] and
the fundamental groups of Seifert fibre spaces [5, p.258][1]. So the adjective ‘faithful’ is not so easily discarded in our definition of a CAT(0) group. We do not know of two abstractly commensurable groups, one of which is CAT(0), and the other of which is not. We promote the following question.

**Question 1.** Is the property of being a CAT(0) group an invariant of the abstract commensurability class of a group?

Some relevant results in the literature show that two natural approaches to this question do not work in general. If $G$ acts geometrically on a CAT(0) space $X$ and $G'$ is a finite extension with $[G' : G] = n$, then $G'$ acts properly and isometrically on the CAT(0) space $X^n$ with the product metric [18][7, p.190]. However, proving this action is cocompact is either difficult or impossible in general. In [2], the authors give examples of the following type: $G$ is a group acting faithfully and geometrically on a CAT(0) space $X$, $G'$ is a finite extension of $G$, yet $G'$ does not act faithfully and geometrically on $X$. However, $G'$ may act faithfully and geometrically on some other CAT(0) space.

**Remark 2.** The fact that $\text{Aut}(F_2)$ is isomorphic to $\text{Aut}(W_3)$ appears to be well-known in certain mathematical circles, but is rarely recorded explicitly. We now outline a proof: the subgroup $E \leq W_3$ of even length elements is isomorphic to $F_2$, characteristic in $W_3$ and $C_{W_3}(E) = \{1\}$; it follows from [17, Lemma 1.1] that the induced homomorphism $\pi : \text{Aut}(W_3) \to \text{Aut}(E)$ is injective; one easily confirms that the image of $\pi$ contains a set of generators for $\text{Aut}(E)$, and hence $\pi$ is an isomorphism. A topological proof may also be constructed using the fact that the subgroup $E$ of even length words in $W_3$ corresponds to the 2-fold orbifold cover of the toroidal orbifold $S^2(2, 2, 2, \infty)$ by the once-punctured torus.

The authors would like to thank Jason Behrstock and Martin Bridson for pointing out the examples in [15] and [5, p.258][1] and Luisa Paoluzzi for discussions regarding [8].

## 2. $\text{Aut}(B_4)$ is a CAT(0) group

We shall describe an apt presentation of $B_4$, give a concise combinatorial description of Brady’s space $X_0$, describe the faithful geometric action of
Inn($B_4$) on $X_0$ and, finally, introduce an isometry of $X_0$ to extend the action of Inn($B_4$) to a faithful geometric action of Aut($B_4$).

The interested reader will find an informative, and rather more geometric, account of $X_0$ and the associated action of Inn($B_4$) in [8].

An apt presentation of $B_4$: A standard presentation of the group $B_4$ is

\[ \langle a, b, c \mid aba = bab, bcb = cbc, ac = ca \rangle. \] (1)

Introducing generators $d = (ac)^{-1}b(ac)$, $e = a^{-1}ba$ and $f = c^{-1}bc$, one may verify that $B_4$ is also presented by

\[ \langle a, b, c, d, e, f \mid ba = ac = eb, de = ec = cd, bc = cf = fb, df = fa = ad, ca = ac, ef = fe \rangle. \] (2)

We set $x = bac$ and write $\langle x \rangle \subset B_4$ for the infinite cyclic subgroup generated by $x$. The center of $B_4$ is the infinite cyclic subgroup generated by $x^4$.

The space $X_0$: Consider the 2-dimensional piecewise Euclidean CW-complex $X_0$ constructed as follows:

(0-S) the vertices of $X_0$ are in one-to-one correspondence with the left cosets of $\langle x \rangle$ in $B_4$—we write $v_{g\langle x \rangle}$ for the vertex corresponding to the coset $g\langle x \rangle$;

(1-S) distinct vertices $v_{g_1\langle x \rangle}$ and $v_{g_2\langle x \rangle}$ are connected by an edge of unit length if and only if there exists an element $\ell \in \{a, b, c, d, e, f\}^{\pm 1}$ such that $g_2^{-1}g_1\ell \in \langle x \rangle$;

(2-S) three vertices $v_{g_1\langle x \rangle}$, $v_{g_2\langle x \rangle}$ and $v_{g_3\langle x \rangle}$ are the vertices of a Euclidean (equilateral) triangle if and only if the vertices are pairwise adjacent.

The link of the vertex $v_{\langle x \rangle}$ in $X_0$, just like the link of each vertex in $X_0$, consists of twelve vertices (one for each of the cosets represented by elements in $\{a, b, c, d, e, f\}^{\pm 1}$) and sixteen edges (one for each of the distinct ways to spell $x$ as a word of length three in the alphabet $\{a, b, c, d, e, f\}$—see [8] for more details). It can be viewed as the 1-skeleton of a Möbius strip. In Figure 1 we depict the infinite cyclic cover of the link of $v_{\langle x \rangle}$. Each vertex with label $g$ in the figure lies above the vertex $v_{g\langle x \rangle}$ in the link of $v_{\langle x \rangle}$. The link is formed by identifying identically labeled vertices and identifying edges with the same start and end points.

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That $X_0$ is CAT(0) follows most naturally from the alternative construction of $X_0$ described in detail in [8]. Alternatively, a complex constructed from isometric Euclidean triangles is CAT(0) if and only if it is simply-connected and satisfies the ‘link condition’ [5, Theorem II.5.4, pp.206]. For a 2-dimensional complex, the link condition requires that each injective loop in the link of a vertex has length at least $2\pi$, where edges in a link are assigned the length of the angle they subtend [5, Lemma II.5.6, pp.207]. It is easily seen that $X_0$ satisfies the link condition because each injective loop in Figure 1 crosses at least 6 edges and each edge has length $\pi/3$. Thus one might show that $X_0$ is CAT(0) by showing that it is simply-connected. We shall not digress from the task at hand to provide such an argument.

**Brady’s faithful geometric action of Inn($B_4$) on $X_0$:** We shall describe Brady’s faithful geometric action of Inn($B_4$) on $X_0$. We shall do so by describing an isometric action $\rho : B_4 \rightarrow \text{Isom}(X_0)$ such that the image of $\rho$ is a properly discontinuous and cocompact subgroup of Isom($X_0$) which is isomorphic to Inn($B_4$).

It follows immediately from (1-S) that, for each $g \in B_4$, the “left-multiplication by $g$” map on the 0-skeleton of $X_0$, $g_1(x) \mapsto gg_1(x)$, extends to a simplicial isometry of the 1-skeleton of $X_0$. It follows immediately from (2-S) that any simplicial isometry of the 1-skeleton of $X_0$ extends to a simplicial isometry of $X_0$. We write $\phi_g$ for the isometry of $X_0$ determined by $g$ in this way, and we write $\rho : B_4 \rightarrow \text{Isom}(X_0)$ for the map $g \mapsto \phi_g$. We compute that $\rho(g_1g_2)(v_{g(x)}) = v_{g_1g_2g(x)} = \rho(g_1)\rho(g_2)(v_{g(x)})$ for each $g_1, g_2, g \in B_4$, so $\rho$ is a homomorphism. Further, $\phi_g(v_{(x)}) = v_{g(x)}$ for each $g \in B_4$, so the vertices of $X_0$ are contained in a single $\rho$-orbit. It follows that $\rho$ is a cocompact isometric action of $B_4$ on $X_0$.

To show that the image of $\rho$ is isomorphic to Inn($B_4$), it suffices to show...
that the kernel of $\rho$ is exactly the center of $B_4$. One easily computes that $\rho(x^4)$ is the identity isometry of $X_0$. Thus the kernel of $\rho$ contains the center of $B_4$. It is also clear that the stabilizer of $v_{(x)}$, which contains the kernel of $\rho$, is the infinite subgroup $\langle x \rangle$. So to establish that the kernel of $\rho$ contains the center of $B_4$, it suffices to show that $\phi_x, \phi_{x^2}$ and $\phi_{x^3}$ are non-trivial and distinct isometries of $X_0$. We achieve this by showing that these elements act non-trivially and distinctly on the link of $v_{(x)}$ in $X_0$. We compute that $x$ acts as follows on the cosets corresponding to vertices in the link of $v_{(x)}$, where $\delta = \pm 1$: 

$$
a^\delta \langle x \rangle \mapsto e^\delta \langle x \rangle \mapsto c^\delta \langle x \rangle \mapsto f^\delta \langle x \rangle \mapsto a^\delta \langle x \rangle \text{ and } b^\delta \langle x \rangle \leftrightarrow d^\delta \langle x \rangle.
$$

Thus the restriction of $\phi_x$ to the link of $v_{(x)}$ may be understood, with reference to Figure 2, as translation two units to the right followed by reflection across the horizontal dotted line. It follows that $\phi_x, \phi_{x^2}, \phi_{x^3}$ are non-trivial and distinct isometries of $X_0$, as required.

We next show that the image of $\rho$ is a properly discontinuous subgroup of $\text{Isom}(X_0)$. Now, the action $\rho$ is not properly discontinuous because, as noted above, the $\rho$-stabilizer of $v_{(x)}$ is the infinite subgroup $\langle x \rangle$ (so infinitely many elements of $B_4$ fail to move the unit ball about $v_{(x)}$ off itself). But the image of $\langle x \rangle$ under the map $B_4 \rightarrow \text{Inn}(B_4)$ has order four. It follows that the image of $\rho$ is a properly discontinuous subgroup of $\text{Isom}(X_0)$.

Thus we have that the image of $\rho$ is a properly discontinuous and cocompact subgroup of $\text{Isom}(X_0)$ which is isomorphic to $\text{Inn}(B_4)$.

**Extending $\rho$ by finding one more isometry:** It was shown in [10] that the unique non-trivial outer automorphism of $B_n$ is represented by the automorphism which inverts each of the generators in Presentation (1). Consider the automorphism $\tau \in \text{Aut}(B_4)$ determined by

$$
a \mapsto a^{-1}, \quad b \mapsto d^{-1}, \quad c \mapsto c^{-1}, \quad d \mapsto b^{-1}, \quad e \mapsto f^{-1}, \quad f \mapsto e^{-1}.
$$

Note that $\tau$ is achieved by first applying the automorphism which inverts each of the generators $a, b$ and $c$ and then applying the inner automorphism $w \mapsto (ac)^{-1}w(ac)$ for each $w \in B_4$. It follows that $\tau$ is an involution which represents the unique non-trivial outer automorphism of $B_4$. Writing $J := B_4 \rtimes \mathbb{Z}_2$, we have $\text{Aut}(B_4) \cong J/\langle x^4 \rangle$. We identify $B_4$ with its image in $J$.

The automorphism $\tau \in \text{Aut}(B_4)$ permutes the elements of $\{a, b, c, d, e, f\}^{\pm 1}$ and maps the subgroup $\langle x \rangle$ to itself (in fact, $\tau(x) = x^{-1}$). It follows from
(1-S) that the map $v_{g_1(x)} \mapsto v_{\tau(g_1)(x)}$ on the 0-skeleton of $X_0$ extends to a simplicial isometry of the 1-skeleton of $X_0$, and hence also to a simplicial isometry $\theta$ of $X_0$. We compute that $\theta \phi_g \theta = \phi_{\tau(g)}$ for each $g \in B_4$. Thus we have an isometric action $\rho': J \rightarrow \text{Isom}(X_0)$ given by

$$g \mapsto \phi_g \text{ for each } g \in B_4, \text{ and } \tau \mapsto \theta.$$  

We also compute that the restriction of $\theta$ to the link of $v_{(x)}$ may be understood as reflection across the vertical dotted line shown in Figure 2. It follows that $\theta$ is a non-trivial isometry of $X_0$ which is distinct from $\phi_x, \phi_{x^2}$ and $\phi_{x^3}$. Thus the kernel of $\rho'$ is still the center of $B_4$, and the image of $\rho'$ is a properly discontinuous and cocompact subgroup of $\text{Isom}(X_0)$ which is isomorphic to $\text{Aut}(B_4)$. Hence we have a faithful geometric action of $\text{Aut}(B_4)$ on $X_0$, as required.

**References**


