Primitives in the Free Group of Rank Two

Adam Piggott
adam.piggott@tufts.edu

http://www.tufts.edu/~apiggo01/research

Tufts University

Notation

Let $F = F(x, y)$ denote the free group of rank two.

Let $\mathcal{A}^*$ denote the set of finite words in the alphabet $\{x, x^{-1}, y, y^{-1}\}$.

Identify F with the set of reduced words in $\mathcal{A}^*$.

Vocabulary

The subset $\{w, v\} \subset$ F is a **basis** if it generates F.

An element $w \in$ F is **primitive** if it is an element of some basis. Elements $w, v \in$ F are said to be **associated primitives** if $\{w, v\}$ is a basis.

An element $w \in$ F is a **palindrome** if it reads the same "forwards and backwards".

$$e.g. \quad xyx^{-2}yx^{-2}yx \quad \text{is a palindrome}$$

Why primitive elements?

There is a one-to-one correspondence between Aut(F) and ordered bases of F.

Studying (ordered) bases (and therefore primitive elements) is one approach to studying Aut(F).

Some observations

(Bardakov, Shpilrain and Tolstykh, 2005)

Each conjugacy class of primitive elements contains an element w such that either xwy^{-1} is a palindrome or $x^{-1}wy$ is a palindrome.

Each primitive element is either a palindrome, or the product of two palindromes.

Our goal today

The observations by B, S and T are glimpses of a link between palindromes and primitives in F. Our goal is to better understand this link.

Along the way we will also consider the question "What does a basis of F look like?".

Facts about Primitive Elements

There exists an algorithm to determine whether or not $w \in \mathrm{F}$ is primitive, and whether or not $\{w, v\} \subset \mathrm{F}$ is a basis (Whitehead).

If $w \in \mathrm{F}$ is primitive then its exponent sum pair is relatively prime. If $\{w, v\} \subset \mathrm{F}$ is a basis then the matrix of exponent sums has determinant ± 1.

Facts about Primitive Elements II

If $w \in \mathrm{F}$ is primitive then so is each conjugate of w.

Each relatively prime pair of integers uniquely determines a conjugacy class of primitive elements (Nielsen).

Constructing Primitive Elements

(Osborne and Zieschang, 1981)

Given a pair of relatively prime integers X and Y such that $0 \leq X \leq Y$, define $f : \mathbb{Z} \rightarrow \mathcal{A}^{\pm}$ by

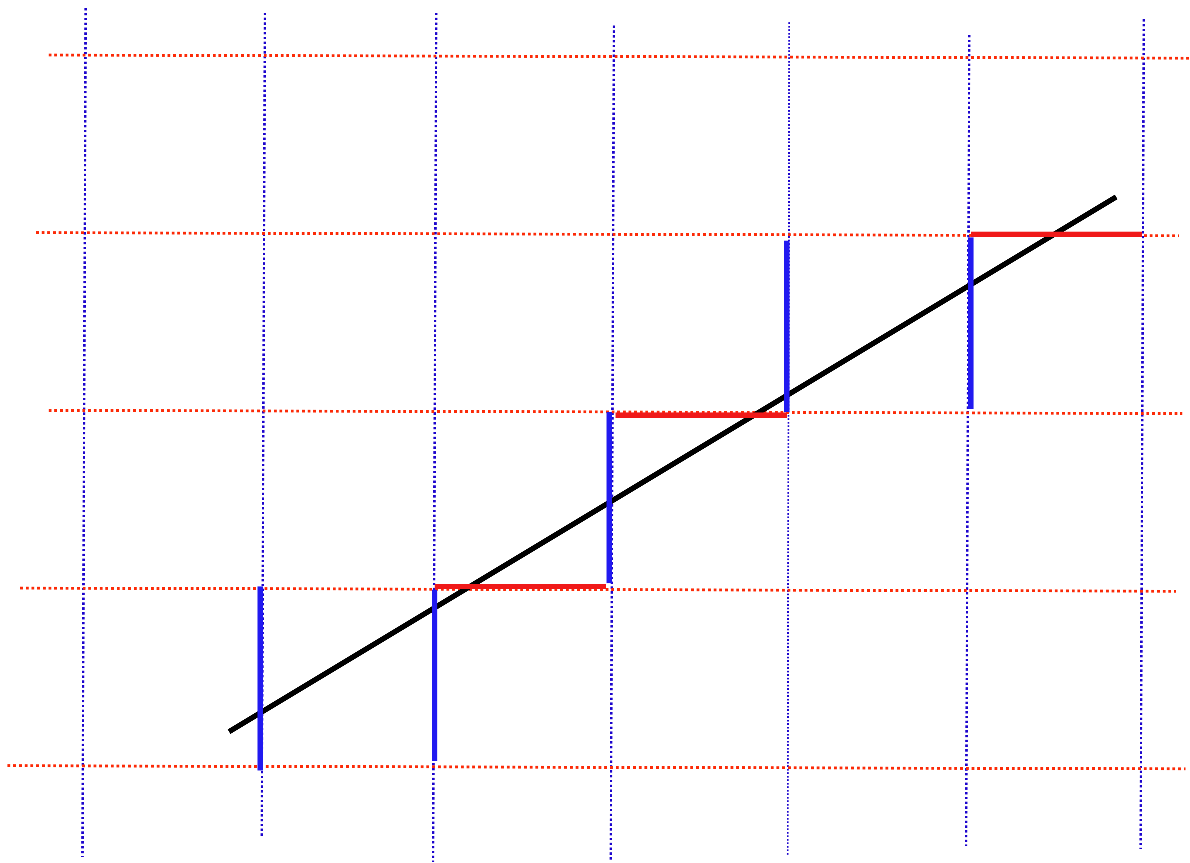
$$f(z) = \begin{cases} x, & \text{if } z \cong 1, \ldots, X \mod (X+Y) \\ y, & \text{if } z \cong X+1, \ldots, X+Y \mod (X+Y) \end{cases}$$

The word

$$f(1)f(1+X)f(1+2X)\ldots f(1+(X+Y-1)X)$$

is primitive with exponent sums X and Y.

Why does O. and Z. construction work?

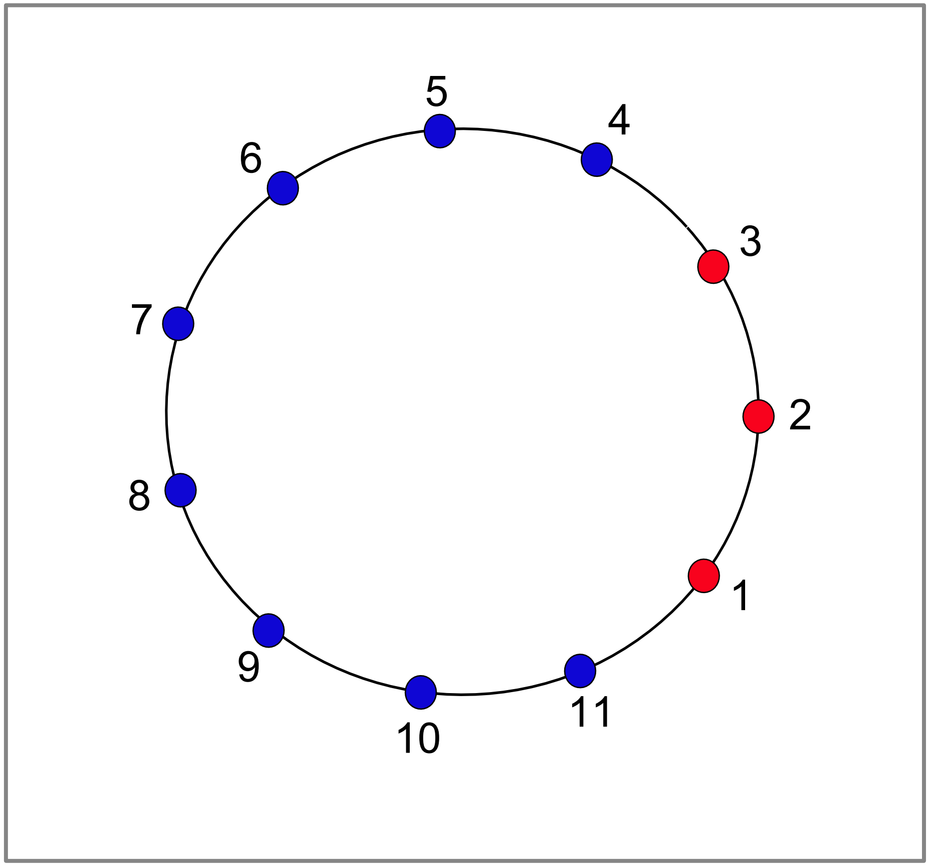


Primitives and palindromes

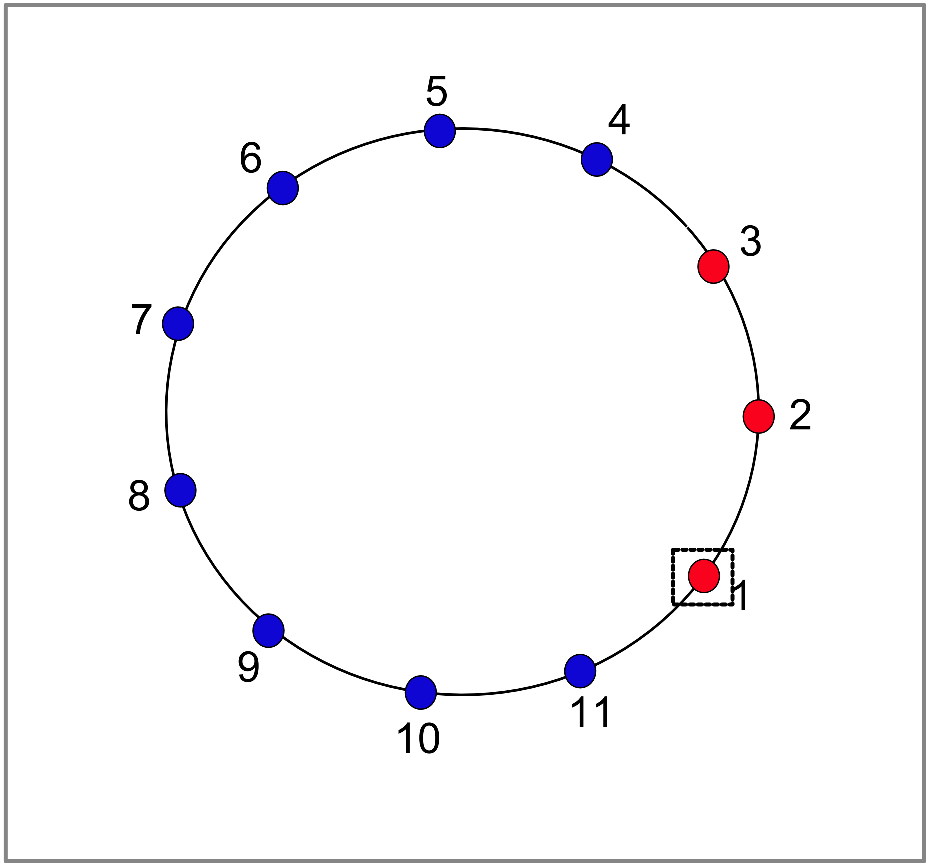
(Kassel and Reutenauer, 2005; P., 2005)
Let p be a primitive element with exponent sums X and Y. Then:

1. $X + Y$ is odd if and only if the conjugacy class of p contains exactly one palindrome (a *palindromic primitive*).

$X = 3, Y = 8$

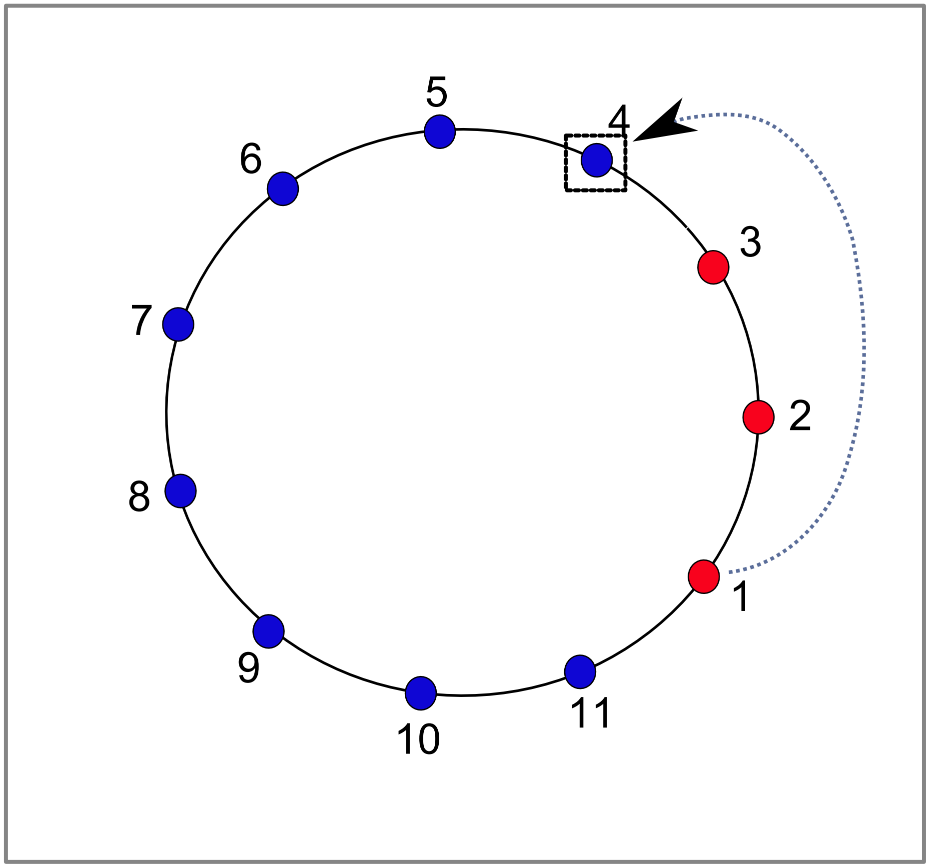


$X = 3, Y = 8$



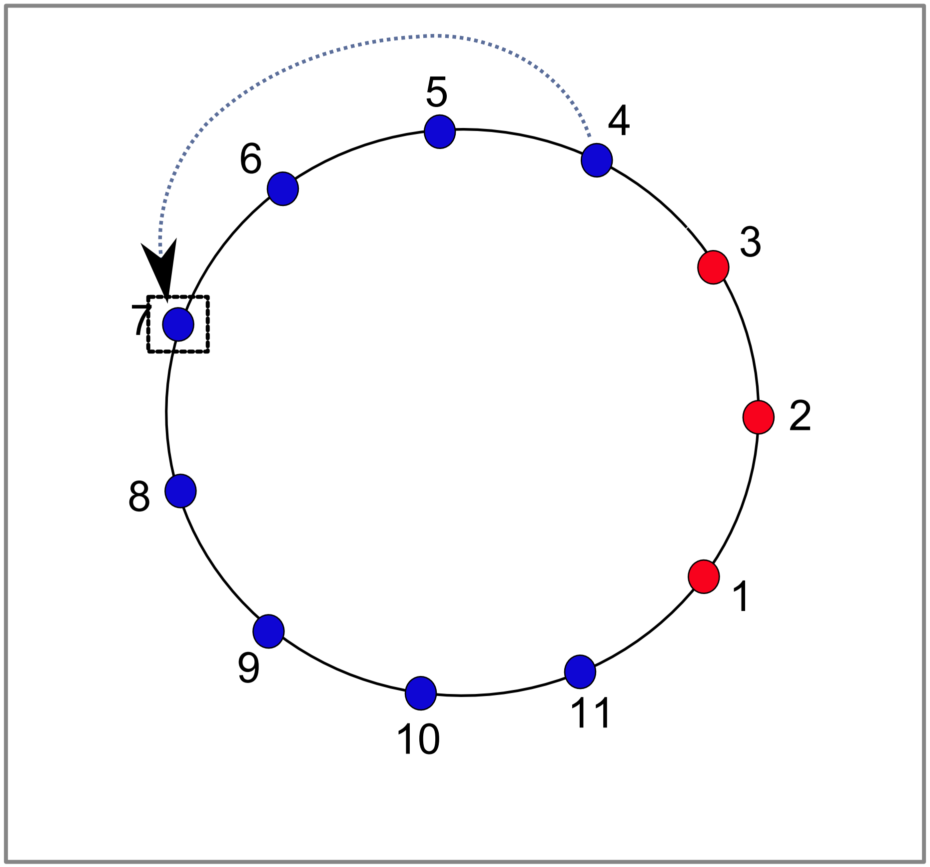
x

$X = 3, Y = 8$



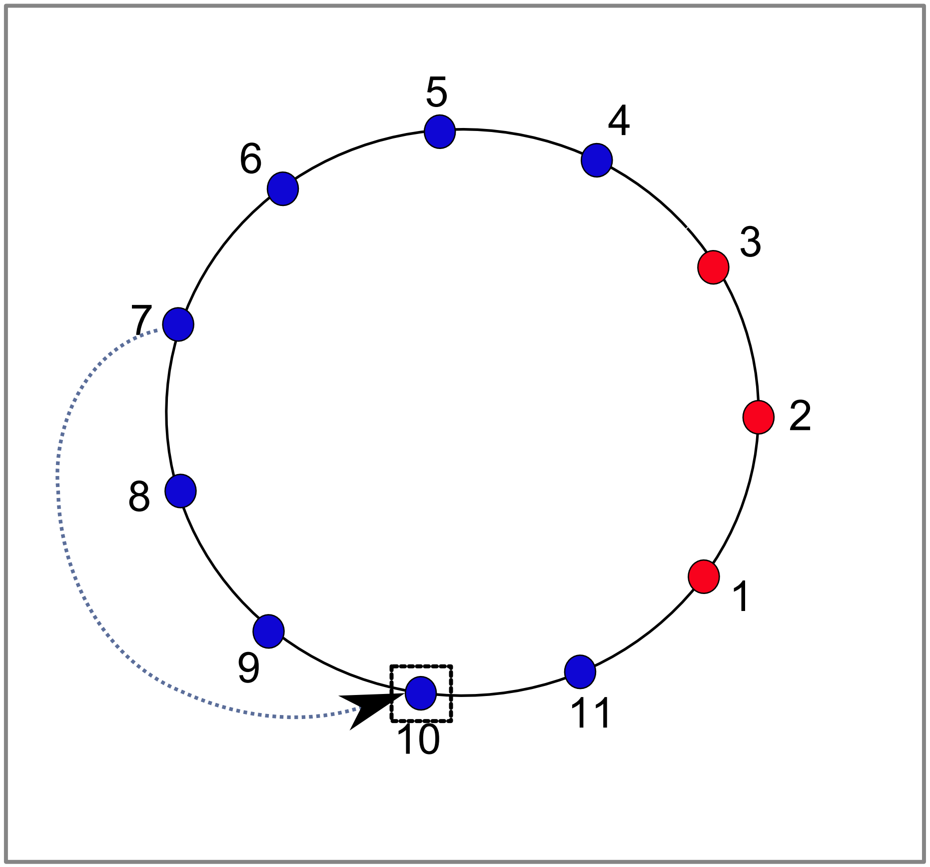
xy

$X = 3$, $Y = 8$



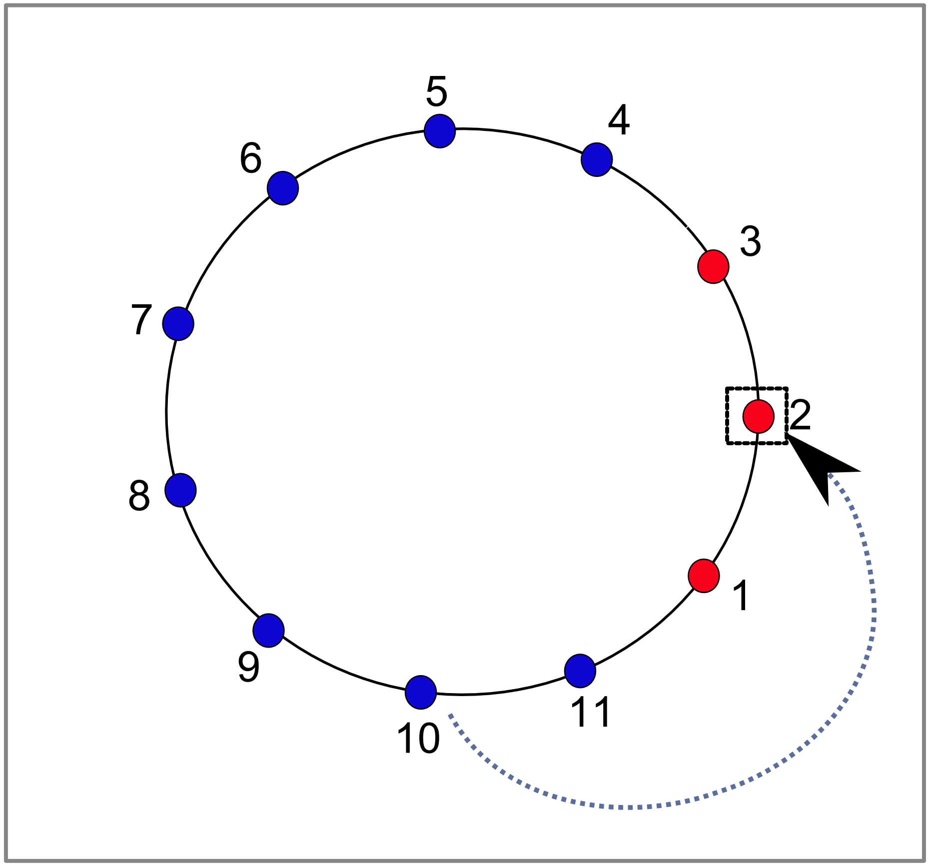
$xy\mathcal{Y}$

$X = 3, Y = 8$



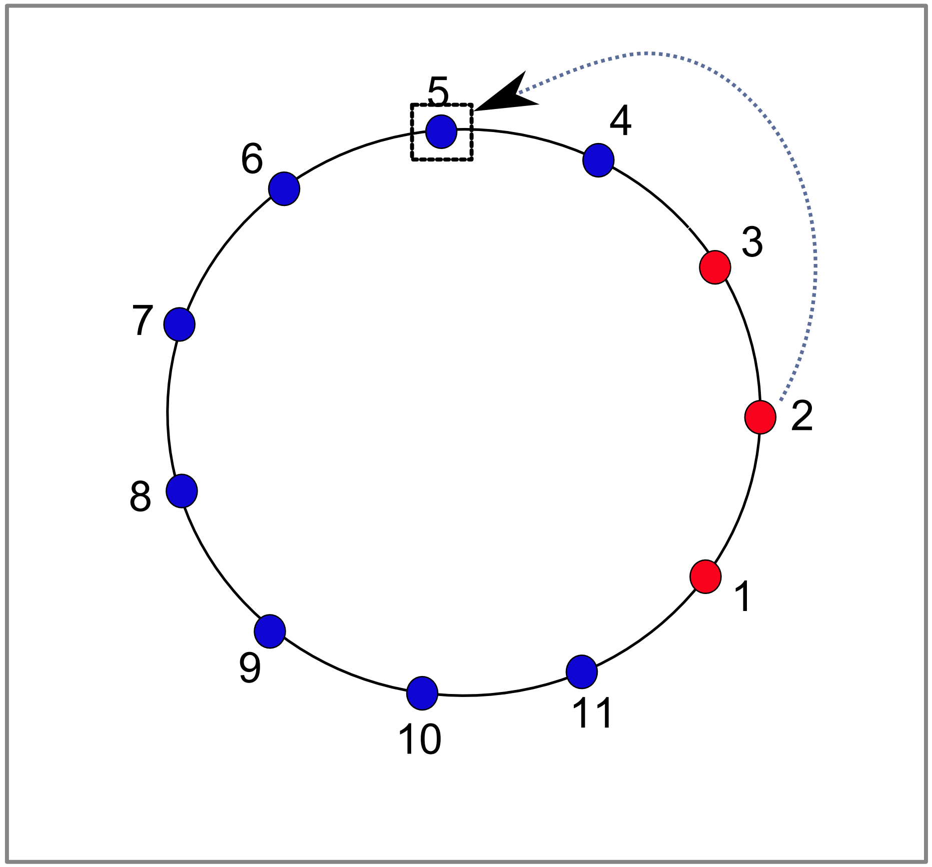
$xyy\mathit{y}$

$X = 3, Y = 8$



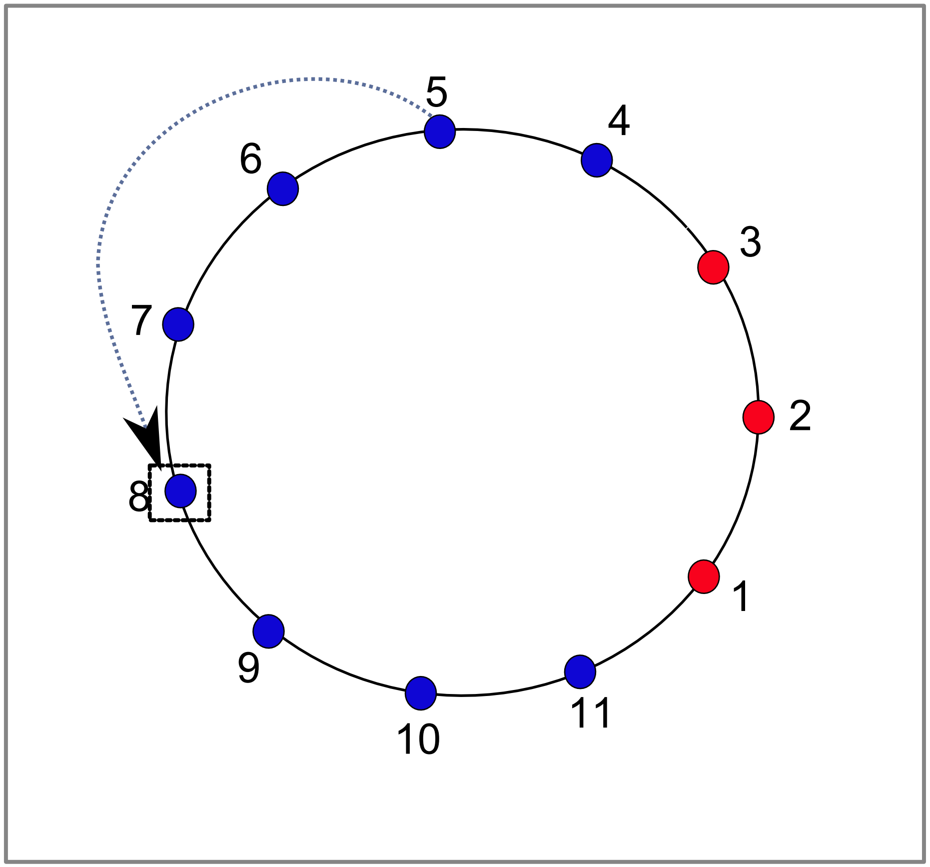
$xyyy\color{red}{x}$

$X = 3, Y = 8$



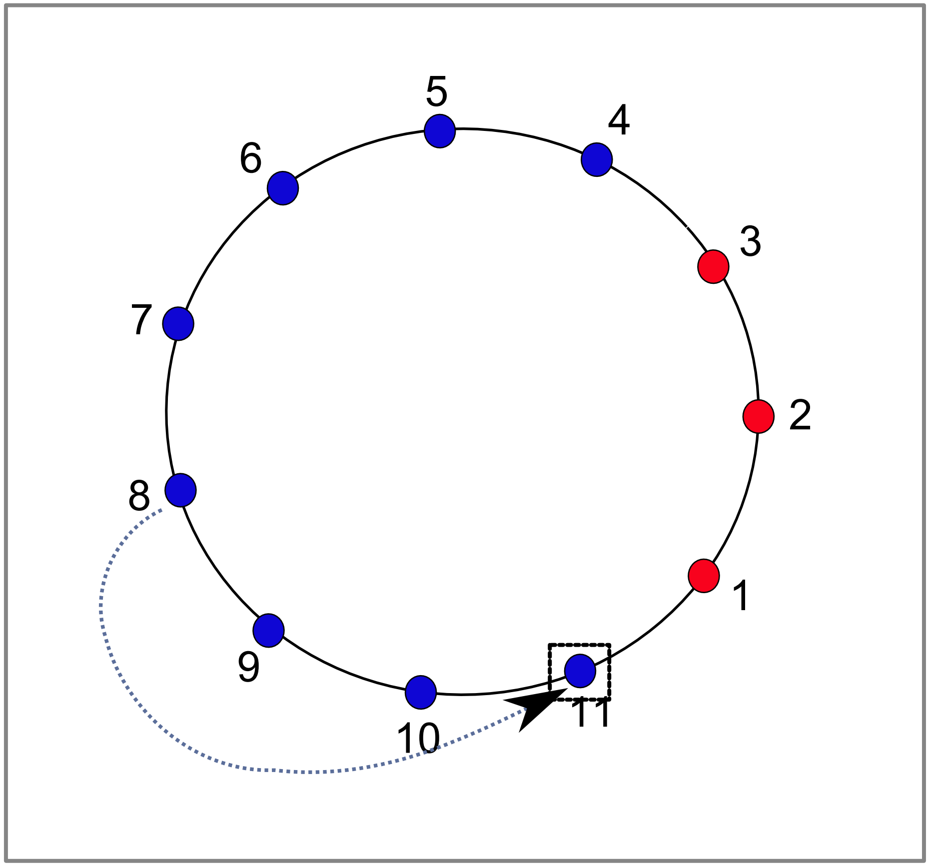
xyyyyxy

$X = 3, Y = 8$



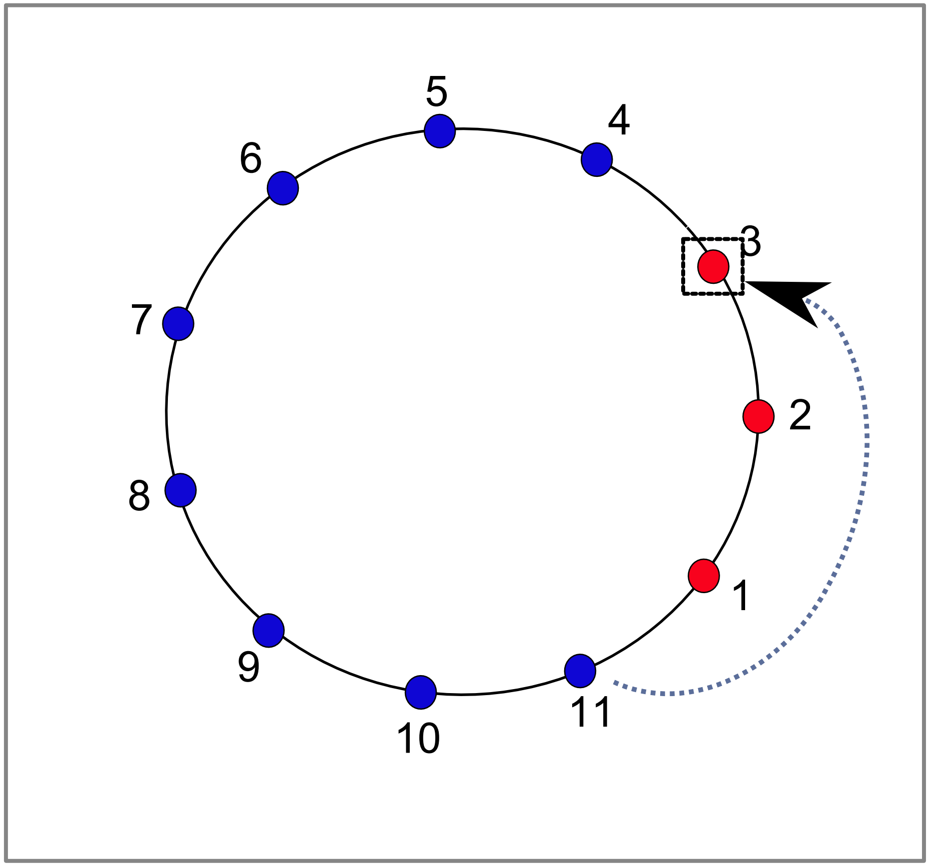
$xyyyxyy$

$X = 3, Y = 8$



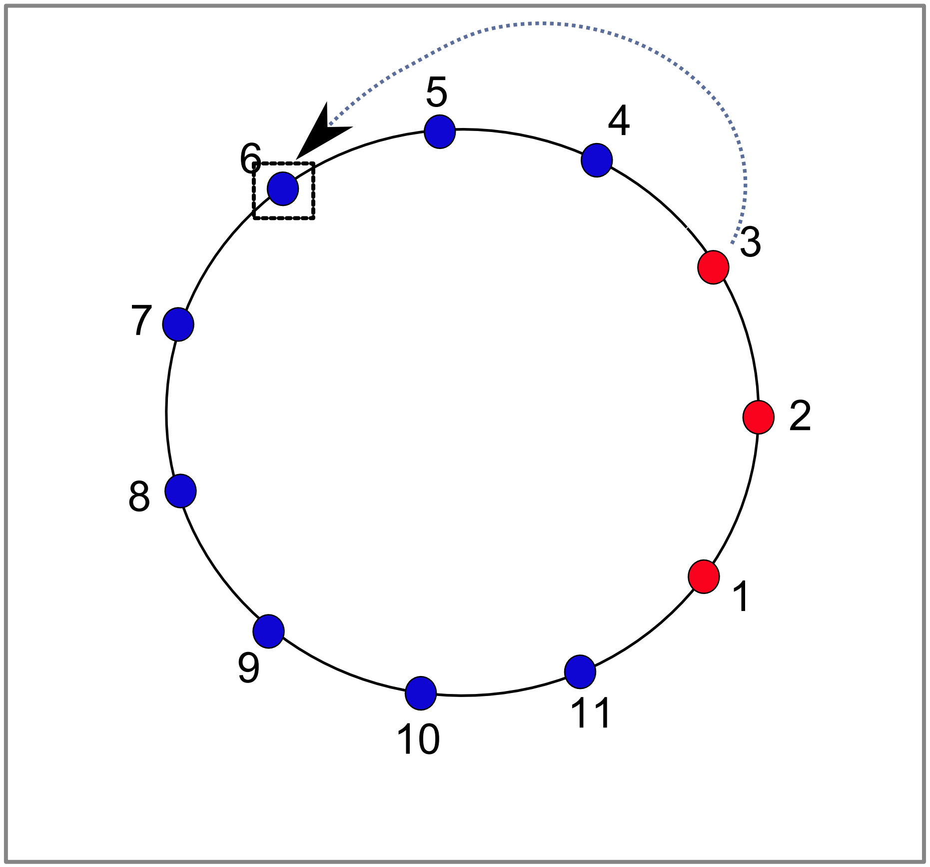
$xyyyxyyy\color{blue}{y}$

$X = 3, Y = 8$



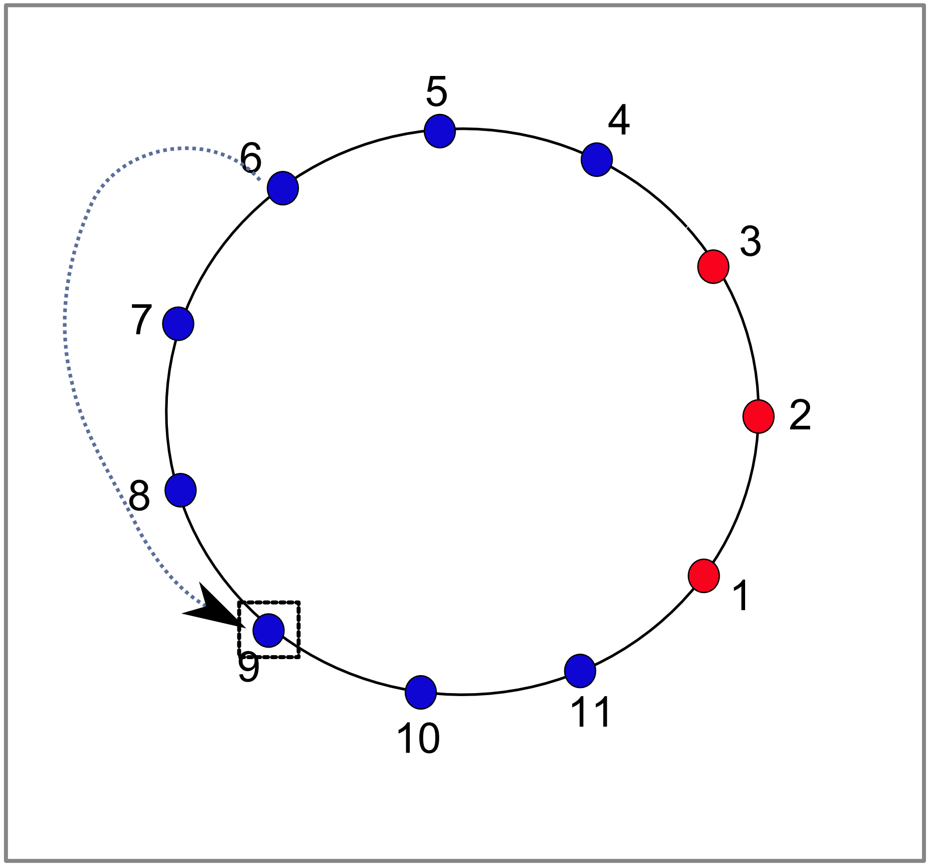
$xyyyxyyyx$

$X = 3, Y = 8$



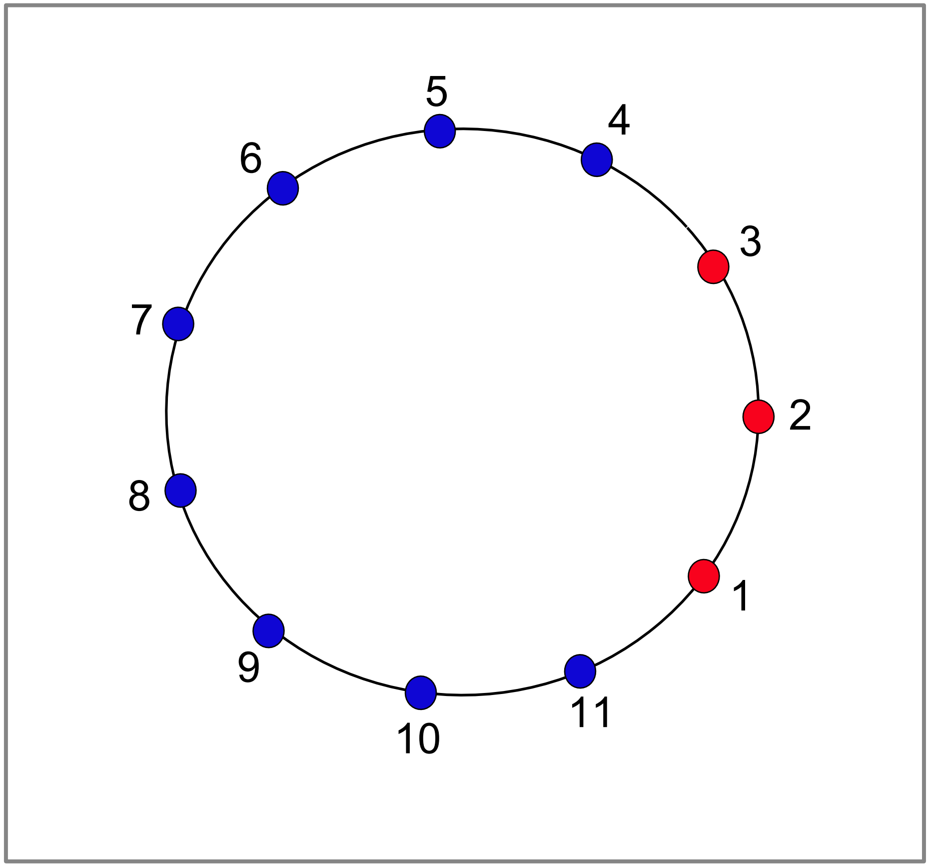
$xyyyxyyyxy$

$X = 3, Y = 8$



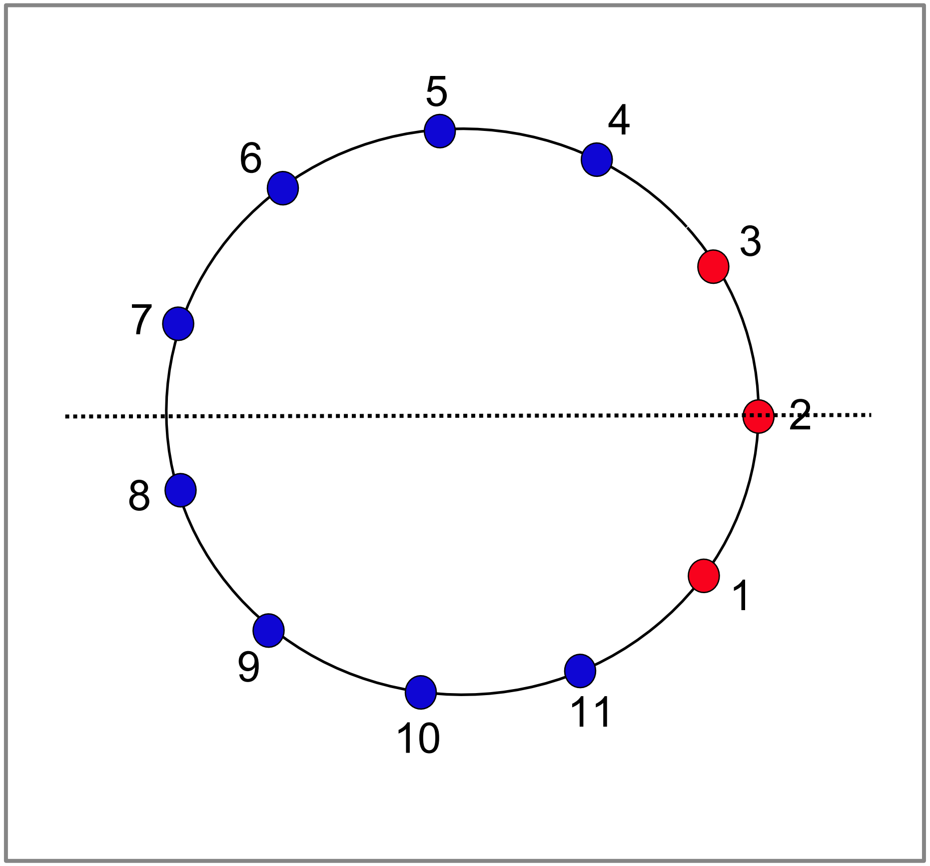
$xyyyxyyyyxy\mathbf{y}$

$X = 3, Y = 8$

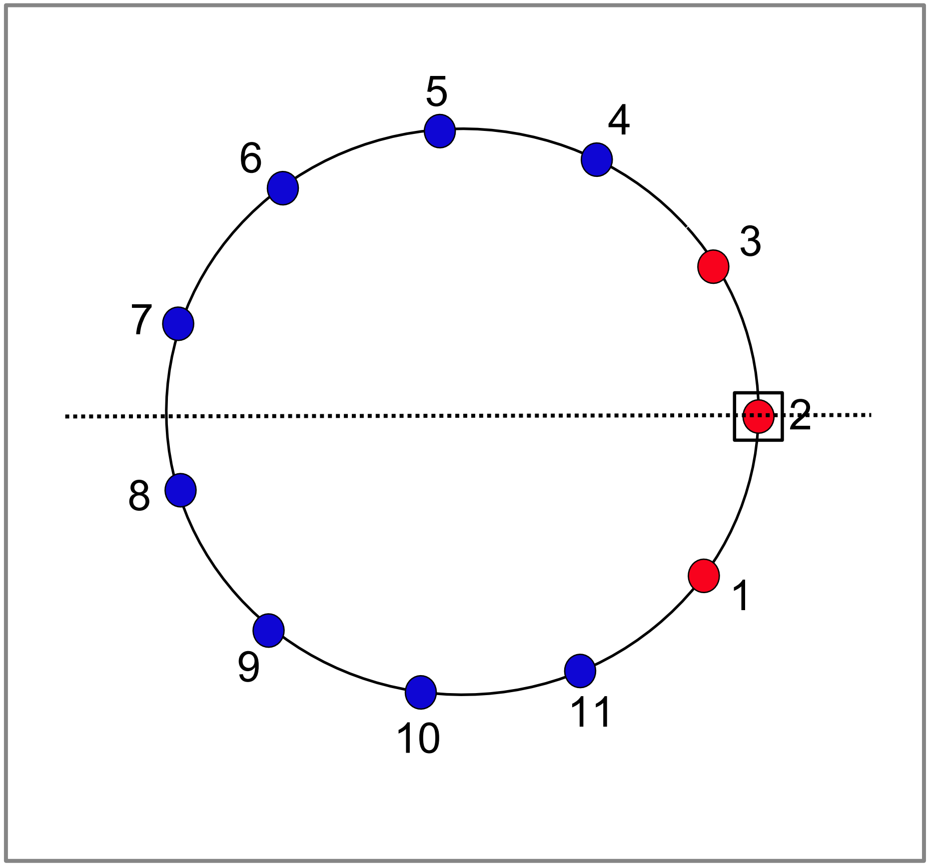


$xyyyxyyyxyy$

$X = 3, Y = 8$

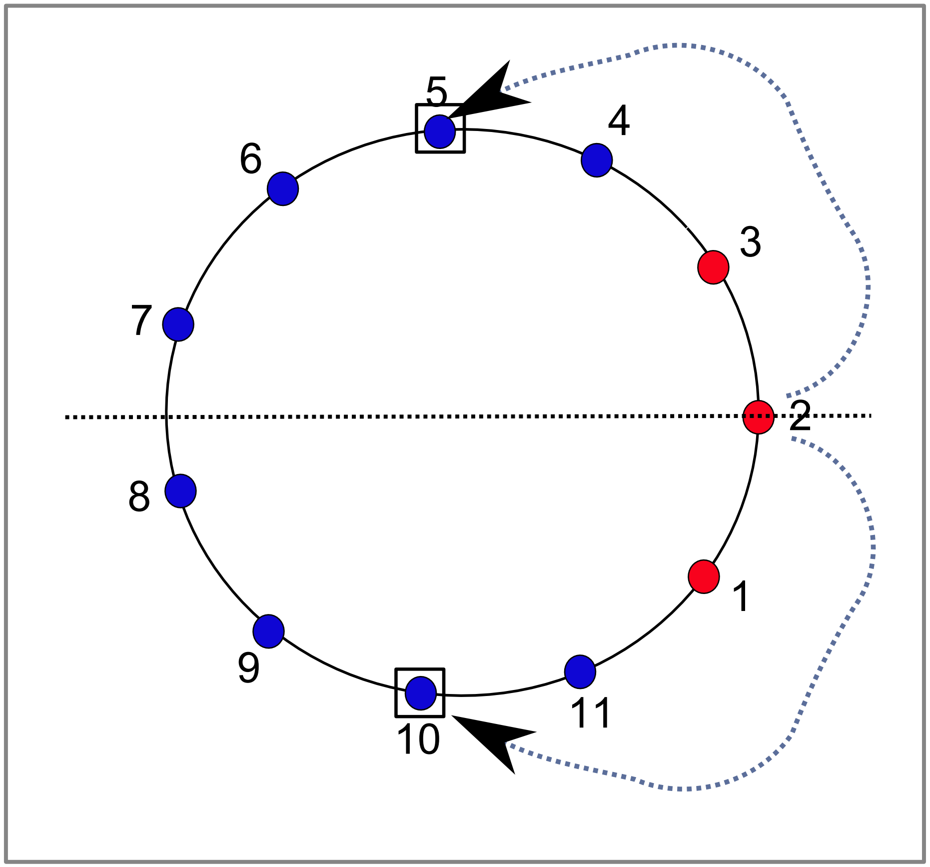


$X = 3, Y = 8$



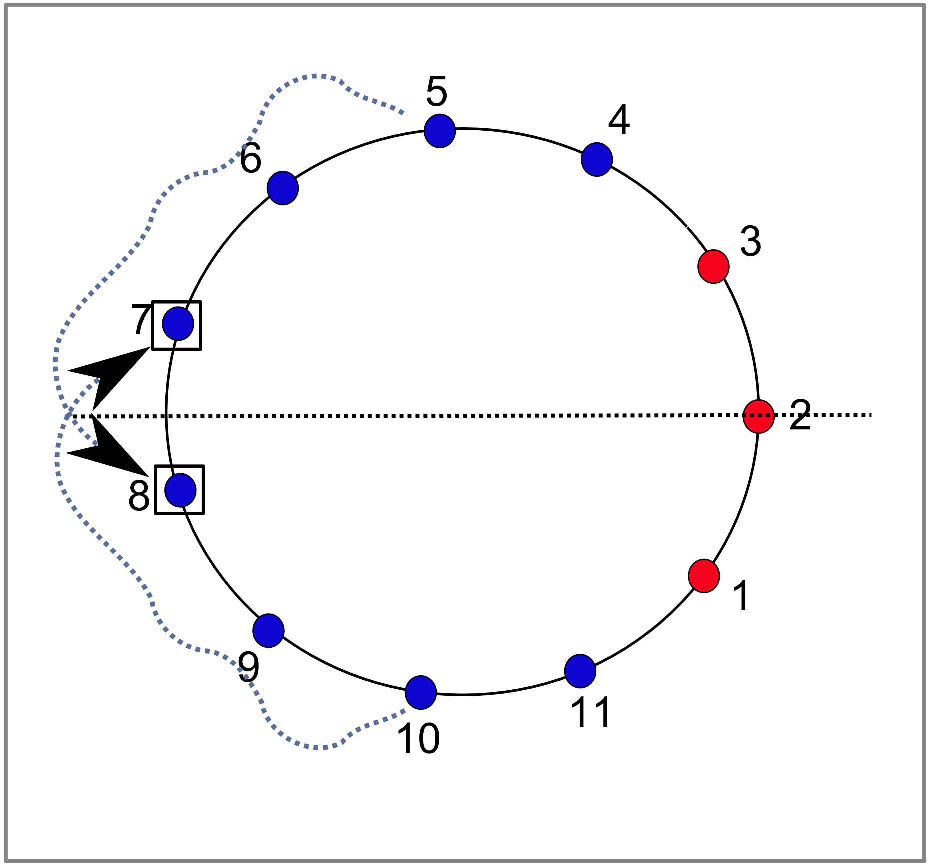
$yxyyyxyyyyxy$

$X = 3, Y = 8$



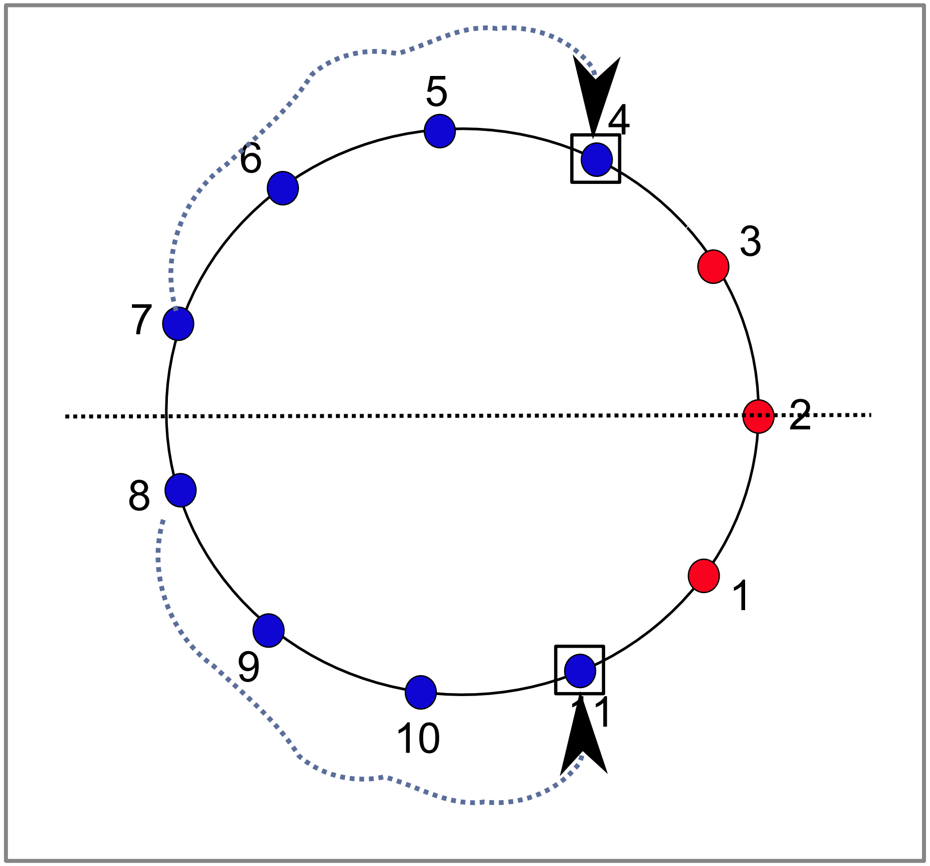
$yxyy\color{blue}{y}\color{black}x\color{blue}{y}\color{black}yyxy$

$X = 3, Y = 8$



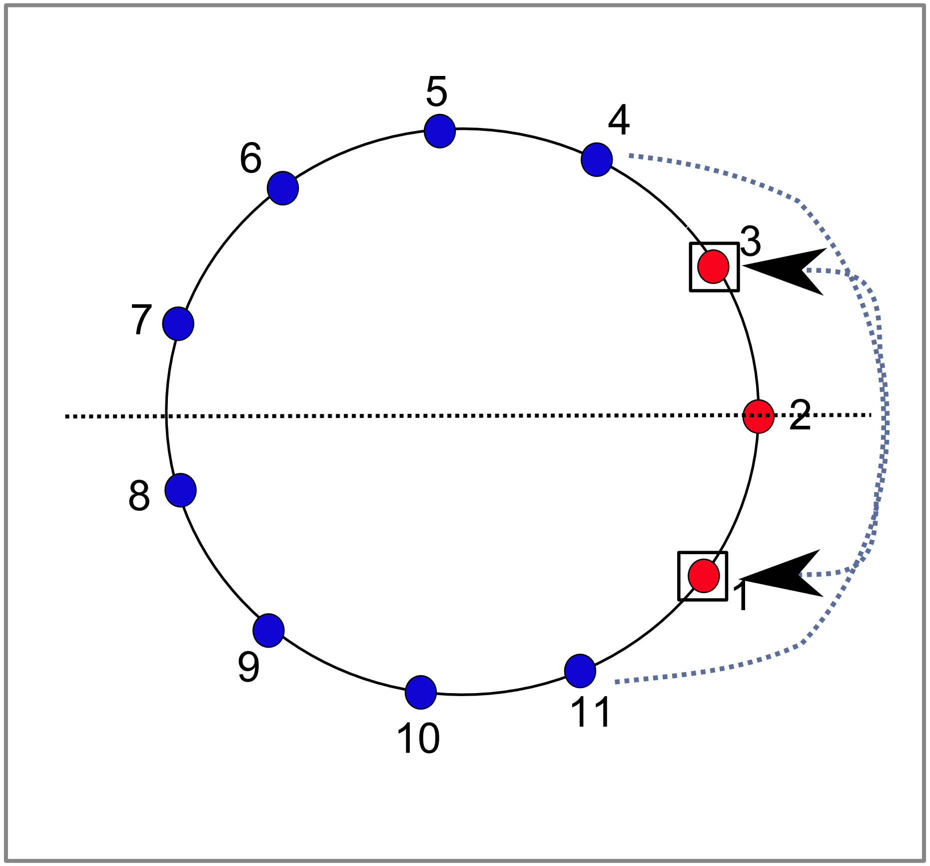
$yxy\mathbf{y}yxy\mathbf{y}yxy$

$X = 3, Y = 8$



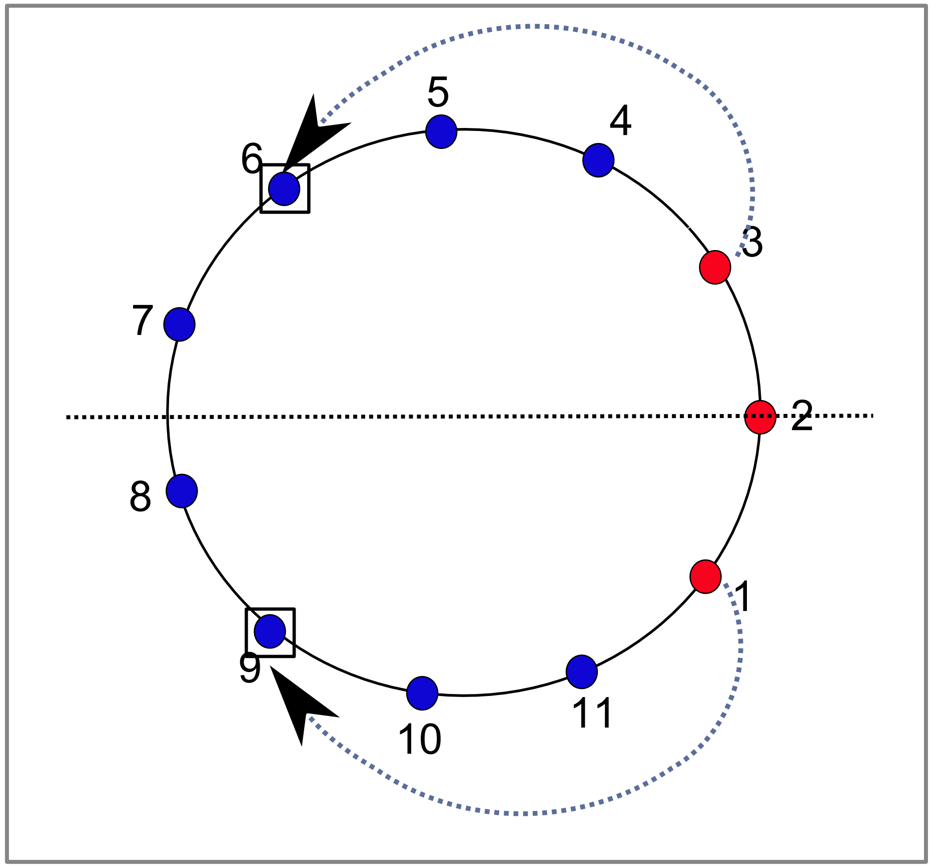
$yx\mathbf{y}yyxyy\mathbf{y}xy$

$X = 3, Y = 8$



$y\textcolor{red}{x}yyyyxyyyy\textcolor{red}{x}y$

$X = 3$, $Y = 8$



$Yyxyyxyyxy Y$

Primitives and palindromes

(Kassel and Reutenauer, 2005; P., 2005)
Let p be a primitive element with exponent sums X and Y. Then:

2. if $X + Y$ is even then the conjugacy class of p does not contain a palindrome but does contain
   (a) exactly one element of the form $x^{\epsilon}w$ $(w.c.)$, and
   (b) exactly one element of the form $y^{\delta}v$ $(w.c.)$,

   where $\epsilon, \delta \in \{\pm 1\}$, the sign of ϵ (resp. δ) matches the sign of X (resp. Y), and $w, v \in \mathrm{F}$ are palindromes of length $|X| + |Y| - 1$.

Palindromic bases

(Kassel and Reutenauer, 2005; P., 2005]

Let A, B, X and Y be integers such that $AY - BX \in \{\pm 1\}$, $A + B$ is odd and $X + Y$ is odd. The unique palindromic primitive p with exponent sum pair (A, B) and the unique palindromic primitive q with exponent sum pair (X, Y) form a basis $\{p, q\}$ of F (a *palindromic basis*).

Concatenation of palindromes

(P., 2005)

For each primitive element $w \in \mathrm{F}$ one of the following holds:

1. w is a palindromic primitive
2. $w \equiv pq$ $(w.c.)$ for non-trivial palindromes p, q
3. $w \equiv apa^{-1}$ $(w.c.)$ for a non-trivial palindrome p and a non-trivial word $a \in \mathrm{F}$
4. $w \equiv apqa^{-1}$ $(w.c.)$ for non-trivial palindromes p, q and a non-trivial word $a \in \mathrm{F}$.

Open question

(Bardakov, Shpilrain and Tolstykh, 2005)

As yet, there is no known algorithm to determine the primitive (or palindromic) length of an element in F (or more generally, F_n). That is, given $w \in \mathrm{F}$, determine minimal s such that $w = p_1 \ldots p_s$ for primitives (or palindromes) $p_1, \ldots, p_s \in \mathrm{F}$.