

A Letter to the Seminario Teoria de Grupos

August 27, 2007

Al Seminario Teoria de Grupos de la Universidad de los Andes:

My friend and collaborator Mauricio Gutierrez asked me to record the following thoughts to provide a discussion point for your seminar. Herein I sketch an argument which proves the rigidity of right-angled Coxeter groups.

The rigidity of right-angled Coxeter groups is a special case of the Even Rigidity Theorem which appears in Michael Laurence's Ph.D. thesis [5] and Patrick Bahls' Ph.D. thesis [1] (see also [2]). It is also a special case of the rigidity of graph products of finite groups, a result of Radcliffe [6]. Arguments proving the rigidity of right-angled Coxeter groups have also appeared in [3] and [4]. Clearly there is no shortage of proofs of this statement. It seems to me that each proof is essentially the same combinatorics, packaged in the way that makes the most sense to the author.

The argument sketched below is yet another way to package the same combinatorics. It relies on a construction, Ω , which nicely encodes the combinatorial properties of W needed to prove the rigidity result. The graph Ω may be found in literature concerning the automorphisms of a right-angled Coxeter group. It is a 'picture' of the groupoid which plays a prominent role in [7], with vertices representing groupoid elements and adjacency recording the domain of the groupoid operation. Your chairman may explain the role of this groupoid in the study of $\text{Aut } W$. I would also like to point out that the main aim of [3] is to compute a presentation of a certain subgroup of $\text{Aut } W$. The subgroup to be presented is none other than $\text{Aut } \Omega$ [7]. So you might say that the main aim of [3] is to describe $\text{Aut } \Omega$ in terms of Γ , although this is not quite the way the author sets about the task.

A similar argument to that below, with suitable generalizations of the constructions, may be used to prove the Even Rigidity Theorem. However,

because of the nature of your endeavor and because so many arguments proving the results involved have already been recorded, I have settled for illustrating the main idea by sketching the argument in the special case of right-angled Coxeter groups.

In the spirit of collective discussion, this letter shall continue in the first person plural.

1 Overview

To a finite undirected graph Γ with vertex set S we associate a group $W(\Gamma)$ with the following presentation:

$$\langle S \mid x^2 = 1 \ (\forall x \in S), \ xy = yx \ (\forall x, y \in S \ x, y \text{ adjacent in } \Gamma) \rangle.$$

The groups which arise in this way are called *right-angled Coxeter groups*. We shall outline a proof of the following well-known result:

Theorem 1.1 (Right-angled Rigidity Theorem). *Let Γ and Γ' be finite graphs. The groups $W(\Gamma)$ and $W(\Gamma')$ are isomorphic if and only if the graphs Γ and Γ' are isomorphic.*

Fix a graph Γ and write $W := W(\Gamma)$. Write $\text{CS}(\Gamma)$ for the set of complete subgraphs of Γ . Let $\Omega = \Omega(W)$ be the graph defined as follows:

1. the vertex set of Ω is $\text{CS}(\Gamma)$;
2. $\Delta, \Theta \in \text{CS}(\Gamma)$ are adjacent in Ω if and only if they are both subgraphs of some complete subgraph $\Lambda \in \text{CS}(\Gamma)$.

The Right-angled Rigidity Theorem is an immediate consequence of the following two lemmas:

Lemma 1.2. *The isomorphism type of Ω is uniquely determined by the isomorphism type of W .*

Lemma 1.3. *The isomorphism type of Γ is uniquely determined by the isomorphism type of Ω .*

2 Lemma 1.2

It suffices to describe an alternate construction of Ω which does not refer to a particular generating set.

Write \mathcal{I} for the set of conjugacy classes of involutions in W . Write Ω' for the graph constructed as follows:

1. the vertex set of Ω' is \mathcal{I} ;
2. conjugacy classes $[u], [v] \in \mathcal{I}$ are adjacent in Ω' if and only if there exist $a \in [u]$ and $b \in [v]$ such that $ab = ba$.

Since the construction of Ω' makes no reference to Γ or S , we immediately have that the isomorphism type of Ω' is uniquely determined by the isomorphism type of W .

We say that an element of W is *cyclically reduced (with respect to S)* if there is no shorter element (in the word-metric with respect to S) in the same conjugacy class. Each conjugacy class of involutions in W contains a unique cyclically reduced element. Each cyclically reduced involution is the product of the vertices in a complete subgraph of Γ . Thus there are one-to-one relationships as follows:

$$\begin{aligned} \text{conjugacy classes of involutions in } W &\leftrightarrow \text{cyclically reduced involutions} \\ &\leftrightarrow \text{CS}(\Gamma). \end{aligned}$$

The relationship between conjugacy classes of involutions in W and $\text{CS}(\Gamma)$ determines an isomorphism $\Omega' \rightarrow \Omega$. We leave the reader to verify that the adjacency relations in Ω' and Ω coincide.

3 Lemma 1.3

It suffices to describe a deterministic procedure to construct Γ using only the data available in Ω . That is, we need to understand how to reconstruct Γ from Ω .

As discussed above, there is a one-to-one correspondence between the complete subgraphs of Γ and the cyclically reduced involutions in W . For a complete subgraph $\Theta \subseteq \Gamma$, we shall write u_Θ for the corresponding cyclically reduced involution. Observe that the star of $\Theta \in \text{CS}(\Gamma)$ in Γ (that is, the set of vertices in Γ that are adjacent to every vertex in Θ) generates the

centralizer of u_Θ . If $\Theta, \Delta \in \text{CS}(\Gamma)$ and $\Theta \subset \Delta$, then the star of Δ in Γ is contained in the star of Θ and u_Θ commutes with more involutions than u_Δ does. The inclusion of stars in Γ determines a partial order on $\text{CS}(\Gamma)$. This partial order is equivalent to the partial order on $\text{CS}(\Gamma)$ determined by the inclusion of stars in Ω . Because complete subgraphs in Γ with ‘more’ vertices are adjacent to ‘fewer’ vertices in Ω , we are able to identify those equivalence classes of vertices of Ω which contains ‘generators’ of Γ and how many generators they must contain.

A deterministic procedure to reconstruct Γ from Ω : We declare vertices in Ω equivalent if they have identical stars. Write V_1, \dots, V_N for the corresponding equivalence classes of vertices. Note that each V_i generates a complete subgraph of Ω . The equivalence relation determines a well-defined quotient graph Ω_0 , with vertices V_1, \dots, V_N and where V_i and V_j are adjacent in Ω_0 if and only if some (and hence all) $v_i \in V_i$ and $v_j \in V_j$ are adjacent.

We shall define a function $f: \{1, \dots, N\} \rightarrow \mathbb{N} \cup \{0\}$ without reference to Γ and construct a graph Γ' from Ω_0 by ‘replacing’ each vertex V_i of Ω_0 by a complete graph with $f(i)$ vertices ($f(i) = 0$ means that V_i is replaced by the graph with no vertices, so we omit V_i and all incident edges). That is, Γ' is defined as follows:

1. for each $1 \leq i \leq N$ such that $f(i) > 0$, let T_i be a complete graph with $f(i)$ vertices;
2. beginning with the disjoint union $\sqcup_{f(i)>0} T_i$, add edges so that each vertex of T_i is adjacent to each vertex of T_j if V_i and V_j are adjacent in Ω_0 .

We write $V_i \preceq V_j$ if the star of V_j (in Ω_0) includes the star of V_i . We write $V_i \prec V_j$ if the inclusion is strict. We say that a strictly ascending chain

$$V_{i_1} \prec V_{i_2} \prec \dots \prec V_{i_k}$$

has length k . We say that V_i has height k if the maximal length strictly ascending chain with $V_i = V_{i_1}$ has length k (so maximal elements have height 1). For each $1 \leq i \leq N$, define

$$\Theta_i := \bigcup_{V_i \preceq V_j} V_j.$$

Define $f: \{1, \dots, N\} \rightarrow \mathbb{N} \cup \{0\}$ as follows:

1. if V_i has height 1, then $f(i) := \log_2(|V_i| + 1)$;
2. inductively, for $h = 2, \dots, N$, if V_i has height h then

$$f(i) := \log_2(|\Theta_i| + 1) - \sum_{V_i \prec V_j} f(j).$$

It is clear that the isomorphism type of Γ' is determined by the isomorphism type of Ω . Thus Lemma 1.3 is proven if one can show that the graphs Γ and Γ' are isomorphic. The details of such an argument are largely left to the reader. We remark only that, if V_i ‘contains’ the vertices corresponding to S_i , then Θ_i corresponds to the intersection of the maximal complete subgraphs of Γ which contain S_i .

Regards
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References

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