ON THE DERIVED LENGTH OF COXETER GROUPS

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Abstract. We characterize certain properties of the derived series of Coxeter groups by properties of the corresponding Coxeter graphs. In particular, we give necessary and sufficient conditions for a Coxeter group to be quasiperfect.

1. Introduction

The abstract groups now known as Coxeter groups were introduced by H.S.M. Coxeter in 1934 as a generalization of Euclidean reflection groups. This class of groups has since been deeply studied, and provides a rich source of examples in group theory and topology. Indeed, since any group generated by involutions is a quotient of a Coxeter group, this class of groups sheds light on many aspects of the general theory of groups.

Each Coxeter group is determined by a labeled graph, called a Coxeter graph. Although non-isomorphic graphs can determine isomorphic groups, many structural properties of Coxeter groups can be characterized in graphical terms. This paper investigates properties of the derived series of (finitely generated) Coxeter groups. In particular, we provide necessary and sufficient graphical conditions for an even Coxeter group to have finite derived length (Section 4), and for any Coxeter group to be quasiperfect (Section 5). We also analyze the derived length of a free product of Coxeter groups (Section 3). Our study builds on recent work of Edjvet and Jeong [8, 4], and is motivated, at least in part, by the fact that the derived series of a group carries information that passes to all of its quotients.

2. Preliminaries

In this section we summarize the necessary background on groups and their derived series, and collect properties of Coxeter groups needed in the proofs of our main results. The reader is referred to [3, Chapter IV] and [6] for further information on Coxeter groups.

2.1. **Derived subgroups.** Let G be a group, and $[G,G] = \langle [x,y] \colon x,y \in G \rangle$ the derived subgroup of G. We write $G^{(0)} := G$ and, for $i \geqslant 1$, $G^{(i)} := [G^{(i-1)}, G^{(i-1)}]$. Then the subgroup chain

$$G = G^{(0)} \geqslant G^{(1)} \geqslant \ldots \geqslant G^{(i)} \geqslant \ldots \tag{1}$$

is called the *derived series* of G. We write $\phi(G) = \infty$ if each term in the derived series is distinct; otherwise $\phi(G) = \min\{i \colon G^{(i)} = G^{(i+1)}\}$. Then G is: *perfect* if $\phi(G) = 0$; *soluble* if there exists $m \geqslant 0$ such that $G^{(m)} = 1$; *perfect-by-soluble* if $\phi(G) < \infty$; and *quasiperfect* (or *perfect-by-abelian*) if $\phi(G) \leqslant 1$.

The next three lemmas will be used in the arguments following. The first two follow immediately from the observation that each epimorphism $G \twoheadrightarrow H$ restricts to an epimorphism $G^{(n)} \twoheadrightarrow H^{(n)}$ for each $n \geqslant 0$. The third is an exercise in [10, p. 197]; see also [8, Example 1.4].

Lemma 2.1. *If H is a homomorphic image of a group G, then $\phi(G) \geqslant \phi(H)$.*

Lemma 2.2. *Let $H \leqslant G$ with $G^{(1)} \leqslant H$. Then the following hold:*

(i) *$\phi(G) = \infty$ if and only if $\phi(H) = \infty$; and*
(ii) *if $\phi(G) < \infty$, then $\phi(H) \leqslant \phi(G) \leqslant \phi(H) + 1$.*

Lemma 2.3. *If A and B are nontrivial abelian groups with $|A| \neq 2$, and $G = A * B$ is the free product of A and B, then $\phi(G) = \infty$.*

2.2. **Finitely presented perfect groups.** Let $\langle S \mid R\rangle$ be a finite group presentation of a group G, and let

$$R_{\mathrm{ab}} = R \cup \{sts^{-1}t^{-1} \colon s, t \in S\}. \tag{2}$$

Then $\langle S \mid R_{\mathrm{ab}}\rangle$ is a presentation of $G_{\mathrm{ab}} = G/G^{(1)}$ (called the *abelianization* of G). The information contained in that presentation may be encoded in an $|S| \times |R|$ matrix A, called the *abelianized relation matrix* of the presentation $\langle S \mid R\rangle$. The rows of A are indexed by S, and each $w \in R$ determines a column whose entry in row s is the total degree of s in w.

The following provides an elementary test for perfectness in a finitely presented group (cf. [5, p. 1] and [11]).

Lemma 2.4. *Let $G = \langle S \mid R\rangle$ be a finitely presented group, and A its abelianized relation matrix. Then G is perfect if and only if the Smith normal form of A is the augmented identity matrix. (That is, if the invariant factors of A are all 1.)*

Underlying this result is the fact that each elementary row and column operation on an abelianized relation matrix corresponds to a sequence of Tietze transformations on the corresponding group presentation.

2.3. **Coxeter groups.** Let n be a non-negative integer, let $I = \{0, 1, \ldots, n-1\}$, and let S be the set of symbols $\{s_i \mid i \in I\}$. A *finite Coxeter presentation* (of rank n) is a group presentation of the form

$$\langle S \mid (s_i s_j)^{m_{ij}} \text{ for } i, j \in I\rangle, \tag{3}$$

where $m_{ij} \in \mathbb{N} \cup \{\infty\}$, $m_{ij} = m_{ji}$ for all $i, j \in I$, and $m_{ij} = 1$ if and only if $i = j$. (We interpret $(s_i s_j)^{\infty}$ as no relation.) A *finitely generated Coxeter group* is a group which admits a finite Coxeter presentation. Each element of S determines a distinct involution in the Coxeter group. We shall henceforth omit the qualifiers “finite” and “finitely generated,” although they shall be assumed throughout.

A Coxeter presentation is neatly encoded in a complete labeled undirected graph, called a *Coxeter graph*: there is one vertex for each generator in the presentation and the edge between vertices s_i and s_j is labeled by m_{ij}. If Γ is a Coxeter graph, we denote the Coxeter group it determines by $W(\Gamma)$. As noted in the introduction, it is possible that non-isomorphic Coxeter graphs determine isomorphic Coxeter groups.

Since each relation in a Coxeter presentation has even length, there is a well-defined homomorphism $W \to \{\pm 1\}$ which sends s_i to -1 for each $i \in I$. The kernel of this map is the *alternating subgroup* of W, denoted W^+. Evidently, $[W\colon W^+] = 2$ and $W^{(1)} \leqslant W^+$. Furthermore, if $I_0 = I \setminus \{0\}$, and $T = \{t_i := s_0 s_i \colon i \in I_0\}$, then

$$W^+ = \langle T \mid t_i^{m_{0i}},\ (t_i t_j^{-1})^{m_{ij}} \text{ for } i, j \in I_0 \rangle. \tag{4}$$

This presentation for W^+ is readily computed using the Reidemeister-Schreier process [3, p. 33]. Note that Lemma 2.2 allows us to draw conclusions about $\phi(W)$ from knowledge of $\phi(W^+)$.

3. Decompositions of Coxeter groups

Let Γ be a Coxeter graph, and $W = W(\Gamma)$. If, by deleting all of the edges in Γ labeled 2, we obtain a graph with more than one connected component, then W is the direct product of subgroups U and V that are also Coxeter groups. In that case, $\phi(W) = \max\{\phi(U), \phi(V)\}$, so it is clear that we need consider only the case that W is directly indecomposable.

Suppose that, by deleting all of the edges of Γ labeled ∞, we obtain a graph with more than one connected component. Then W is the free product of subgroups U and V that are also Coxeter groups. It follows from the Torsion Theorem for Free Products [9, Chap. IV, Section 1, Theorem 1.6] that the existence of such a splitting is recognizable in this way from *any* Coxeter graph for W. One consequence of the next result is that, when considering the derived length of a Coxeter group, we need only consider those groups that are free-indecomposable.

Theorem 3.1. *Let U and V be a nontrivial Coxeter groups, and let $W = U * V$ be the free product of U and V. Then $\phi(W) < \infty$ if and only if U and V are quasiperfect, and $[U\colon U^{(1)}] = [V\colon V^{(1)}] = 2$. Furthermore, if the latter conditions are satisfied, then $\phi(W) = 2$.*

Proof. Let $W = U * V$, and suppose that $\phi(W) < \infty$. In view of the natural homomorphism from W onto $(U/U^{(1)}) * (V/V^{(1)})$, it follows immediately from Lemma 2.1 and Lemma 2.3 that $[U\colon U^{(1)}] = [V\colon V^{(1)}] = 2$.

Let Γ and Δ be Coxeter graphs for U and V respectively. Let I be an indexing set for the vertices of Γ, and K an indexing set for the vertices of Δ. Let m_{ij} $(i, j \in I)$ denote the edge labels in Γ, and $n_{k\ell}$ $(k, \ell \in K)$ denote the edge labels in Δ. If $S_U = \{u_i \colon i \in I\}$ and $S_V = \{v_k \colon k \in K\}$, then

$$W = \langle S_U \cup S_V \mid (u_i u_j)^{m_{ij}},\ (v_k v_\ell)^{n_{k\ell}}, \text{ for } i, j \in I,\ k, \ell \in K \rangle. \tag{5}$$

Now, suppose that $\phi(U) > 1$. Put $I_0 := I \setminus \{0\}$, let $T_U = \{a_i \colon i \in I_0\}$, and $T_V = \{b_k \colon k \in K\}$. Then, as in Equation (4), we have

$$W^+ = \langle T_U \cup T_V \mid a_i^{m_{0i}},\ (a_i a_j^{-1})^{m_{ij}},\ (b_k b_\ell^{-1})^{n_{k\ell}}, \text{ for } i, j \in I_0,\ k, \ell \in K \rangle. \tag{6}$$

It is evident from this presentation that $W^+ = X * Y$, where

$$X = \langle T_U \mid a_i^{m_{0i}},\ (a_i a_j^{-1})^{m_{ij}}, \text{ for } i, j \in I_0 \rangle, \text{ and} \tag{7}$$

$$Y = \langle T_V \mid (b_k b_\ell^{-1})^{n_{k\ell}}, \text{ for } k, \ell \in K \rangle. \tag{8}$$

Observe that $X = U^+$. Since $[U\colon U^{(1)}] = 2$, it follows that $X = U^{(1)}$. Furthermore, since $\phi(U) > 1$, $X/X^{(1)}$ is nontrivial.

Next consider the presentation for Y in Equation (8). Observe that, if $|K| > 1$, one may replace the last row of the abelianized relation matrix for the presentation with the sum of all of its rows to create a zero row. If $|K| = 1$, on the other hand, then Y is clearly infinite cyclic. In either case, $Y/Y^{(1)}$ is infinite.

In view of the natural homomorphism from W^+ onto $(X/X^{(1)}) * (Y/Y^{(1)})$, it follows from Lemma 2.1 and Lemma 2.3 that $\phi(W^+) = \infty$. Applying Lemma 2.2(i) to $W^+ \leqslant W$, we see that $\phi(W) = \infty$, which contradicts our original supposition. It follows that $\phi(U) = 1$. Similarly, $\phi(V) = 1$.

Conversely, suppose that $\phi(U) = \phi(V) = 1$, and that $[U\colon U^{(1)}] = [V\colon V^{(1)}] = 2$. Put $D_\infty := \langle x, y \mid x^2,\ y^2\rangle$, the infinite dihedral group. The map $u_i \mapsto x$, $v_k \mapsto y$ $(i \in I,\ k \in K)$ extends to a homomorphism $W \to D_\infty$. Thus $\phi(W) \geqslant 2$.

Since $U^{(1)} = U^{(2)}$ and $U^{(2)} \leqslant W^{(2)}$, we have $W^{(2)}u_i = W^{(2)}u_j$ for all $i, j \in I$. Similarly, $W^{(2)}v_k = W^{(2)}v_\ell$ for all $k, \ell \in K$. Let $g, h \in W^{(1)}$. Then $w = [g, h] \in W^{(2)}$ is a product of commutators of products of commutators in W. Working in $W/W^{(2)}$, one may replace each occurrence of u_i in $W^{(2)}w$ by u_0, and each occurrence of v_k with v_0. Since $\langle u_0, v_0\rangle \cong D_\infty$, and $D_\infty^{(2)} = 1$, it follows that $W/W^{(2)} \cong D_\infty$. In a similar way, we see that $W/W^{(3)} \cong D_\infty$. However, since $W/W^{(2)}$ is a quotient of $W/W^{(3)}$, and each proper quotient of D_∞ is finite, it follows that $W^{(2)} = W^{(3)}$. Hence $\phi(W) = 2$, as required. □

4. EVEN COXETER GROUPS

An *even Coxeter graph* has edge labels in $2\mathbb{N} \cup \{\infty\}$. Such Coxeter graphs form a subclass of Coxeter graphs, which is distinguished in a number of ways. For instance, even Coxeter graphs determine isomorphic groups if and only if they are isomorphic labeled-graphs [1, Theorem 3.9]. Also, presuming that only one of the relations $(s_is_j)^{m_{ij}}$ and $(s_js_i)^{m_{ji}}$ is included in the Coxeter presentation given in Equation (3), then it is precisely the presentations determined by even Coxeter graphs that are efficient (in the sense of [2]).

An *even Coxeter group* is a Coxeter group corresponding to an even Coxeter graph. In this section we prove that an even Coxeter group is perfect-by-soluble if and only if it is soluble, and we completely describe the even Coxeter groups that are soluble. Our results follow rather quickly from the following lemmata, extracted from [8]. It is convenient to interpret $\frac{1}{\infty}$ as 0 and to define $\gcd\{d, \infty\} := d$ for each $d \in \mathbb{N} \cup \{\infty\}$. We assume the notation of Section 2.3.

Lemma 4.1. *Let* $W = \langle x, y, z \mid x^2,\ y^2,\ z^2,\ (xy)^\ell,\ (xz)^m,\ (yz)^n\rangle$. *Then* $\phi(W) < \infty$ *if and only if* ℓ, m, n *are pairwise coprime, or* $\frac{1}{\ell} + \frac{1}{m} + \frac{1}{n} \geqslant 1$.

Lemma 4.2. *Let* Γ *be either of the Coxeter graphs in Figure 1, and let* $W = W(\Gamma)$. *Then* $\phi(W) = \infty$.

It follows from Coxeter's classification of the finite Coxeter groups that a directly-indecomposable even Coxeter group is finite if and only if it is a cyclic group of order two, or a finite dihedral group [3, Chapter VI, Section 4.1]. We now apply Jeong's results above to show that the only other directly-indecomposable even Coxeter

FIGURE 1. Two even Coxeter graphs of rank four for which $\phi(W) = \infty$.

groups which are perfect-by-soluble are the infinite dihedral group, and the group determined by the graph with three vertices and edges labeled 4, 4 and 2. The latter group is the affine Weyl group $\widetilde{B}_2$ [3, p. 211], which is soluble of derived length 3.

Theorem 4.3. *Let W be an even Coxeter group such that $\phi(W) < \infty$. Then $\phi(W) \in \{1, 2, 3\}$, and $W = W_1 \times W_2 \times \ldots \times W_m$, where, for $1 \leqslant i \leqslant m$, W_i is an even Coxeter group isomorphic to one of the following:*

1. *the cyclic group of order 2;*
2. *a dihedral group; or*
3. *the affine Weyl group $\widetilde{B}_2$.*

Proof. Let Γ be an even Coxeter graph, whose vertex set is indexed by I. Suppose that $W = W(\Gamma)$ is directly indecomposable, and $\phi(W) < \infty$. If $|I| = 1$, then W is cyclic of order 2, and if $|I| = 2$, then W is dihedral. We therefore assume that $n = |I| \geqslant 3$. We must prove that W is isomorphic to the group in (3).

Let $S = \{s_i \colon i \in I\}$ be the usual set of generators for W. For each subset $J \subseteq I$, let $T_J = \{t_j : j \in J\}$, $R_J = \{(t_i t_j)^{m_{ij}} : i, j \in J\}$, and $W_J = \langle T_J \mid R_J \rangle$. Then the map sending $s_i \mapsto t_i$ if $i \in J$, and $s_i \mapsto 1$ if $i \in I \setminus J$, extends to an epimorphism $W \twoheadrightarrow W_J$. Thus, by Lemma 2.1, $\phi(W_J) < \infty$ for each subset $J \subseteq I$.

By Lemma 4.1, if $i, j, k \in I$ are distinct and $m_{ij} > 4$, then, since all edge labels are even, it follows that $m_{ik} = m_{jk} = 2$. Hence, if $i, j \in I$ with $m_{ij} > 4$, then $m_{ik} = m_{jk} = 2$ for each $k \in I \setminus \{i, j\}$. Then $W_{\{i,j\}}$ is a factor in a nontrivial direct product decomposition of W, contradicting the indecomposability of W. Hence, each edge label in Γ is either 2 or 4.

Suppose that $i, j, k \in I$ are distinct. By Lemma 4.1, the list m_{ij}, m_{ik}, m_{jk} contains at most two 4's. (Each triangle in Γ contains at most two edges with label 4.)

Suppose $i, j, k, \ell \in I$ are distinct. If at least three of the edges connecting the vertices in $\{i, j, k, \ell\}$ are labeled 4, then $W_{\{i,j,k,\ell\}}$ maps onto one of the Coxeter groups corresponding to a graph in Figure 1, which, in view of Lemma 4.2, is impossible. (Each tetrahedron in Γ contains at most two edges with label 4.)

Since W is indecomposable, the graph obtained from Γ by deleting all edges labeled 2 is connected. It is clear from the preceding paragraph, however, that this is impossible if $|I| > 3$. Hence $|I| = 3$, and Γ has two edges labeled 4, and one labeled 2. The result now follows. $\square$

It is worth noting that Theorem 4.3 does not hold for Coxeter groups in general. Indeed Jeong [8] gives examples of Coxeter groups of derived length 4, and Edjvet and Jeong [4] give examples having derived length 5. One is led to ask whether, for each positive integer k, there is a Coxeter group having derived length k, but the results of Edjvet and Jeong represent all that is known in this direction.

We conclude this section by recording two useful consequences of Theorem 4.3.

Corollary 4.4. *An even Coxeter group is perfect-by-soluble if and only if it is soluble.*

Corollary 4.5. *An even Coxeter group is quasiperfect if and only if it is an elementary abelian 2-group.*

5. Quasiperfect Coxeter groups

Let Γ be a Coxeter graph, whose vertex set is indexed by $I = \{0, \dots, n-1\}$. Let $W = W(\Gamma)$ be the associated Coxeter group, presented on the generating set $S = \{s_i \colon i \in I\}$. Since $W^{(1)} \leqslant W^+ < W$, it is clear that a Coxeter group is never perfect. In this section we present a graphical characterization of when W is quasiperfect.

For each $d \in \mathbb{N}$, we write $\Gamma^{(d)}$ for the graph obtained from Γ by deleting all edges whose labels are divisible by d. By convention ∞ is divisible by all numbers d; thus an edge labeled ∞ appears in no graph $\Gamma^{(d)}$.

Generators s_i and s_j are conjugate in W if and only if they lie in the same connected component of $\Gamma^{(2)}$ [3, p. 5]. It follows that $[W \colon W^{(1)}] = 2^\gamma$, where γ is the number of connected components of $\Gamma^{(2)}$. In particular, $W^+ = W^{(1)}$ if and only if $\Gamma^{(2)}$ is connected. The first result determines precisely when W is quasiperfect in this special case. The result is proved by Jeong in his Ph.D. dissertation [7], but we include a short proof here for completeness.

Proposition 5.1. *Let $W = W(\Gamma)$ be a Coxeter group such that $\Gamma^{(2)}$ is connected. Then W^+ is perfect (whence W is quasiperfect) if and only if $\Gamma^{(p)}$ is connected for all primes p.*

Proof. Suppose that, for some prime $p \geqslant 3$, $\Gamma^{(p)}$ is not connected. Let $J \subseteq I$ be the vertex indices of some connected component of $\Gamma^{(p)}$, and put $K = I \setminus J$. Then, for all $j \in J$ and $k \in K$, m_{jk} is divisible by p. Hence, the map $s_j \mapsto x$, $s_k \mapsto y$, where $j \in J$ and $k \in K$, extends to an epimorphism from W onto the dihedral group $D_{2p} = \langle x, y \mid x^2, y^2, (xy)^p \rangle$, of order $2p$. It follows from Lemma 2.1(ii) and Lemma 2.2, that $\phi(W) \geqslant \phi(D_{2p}) = 2$.

Conversely, suppose that $\Gamma^{(p)}$ is connected for all primes p.

Let A denote the abelianized relation matrix of the presentation for W^+ given in Equation (4). Observe that each labeled edge of Γ defines a column of A: each edge $\{s_0, s_i\}$ $(i \in I_0)$ defines a column having m_{0i} in row i and 0 in all other rows; each edge $\{s_i, s_j\}$ $(i, j \in I_0)$ defines a column having m_{ij} in column i and $-m_{ij}$ in column j. It suffices (by Lemma 2.4) to show that A has Smith normal form equal to the augmented identity matrix.

As described in [11], we proceed by showing that all determinantal divisors of A are equal to 1. (Recall that the k^{th} *determinantal divisor* of a matrix is the greatest common divisor of the determinants of all $k \times k$ submatrices.) Fix a prime p, and an integer $1 \leqslant k \leqslant n-1$. It suffices to exhibit a $k \times k$ submatrix of A whose determinant is not divisible by p.

Since $\Gamma^{(p)}$ is connected, there exists a connected acyclic subgraph, T, of $\Gamma^{(p)}$ on $k+1$ vertices $\{s_0, s_{i_1}, \dots, s_{i_k}\} \subseteq S$. Evidently, if $\{s_i, s_j\}$ is an edge in T, then p does not divide m_{ij}. Consider the $k \times k$ matrix, B, of A, with rows $\{i_1, \dots, i_k\}$, and columns determined by the k edges of T. One verifies that a suitable permutation of rows converts B into an upper-triangular matrix whose diagonal entries are $\pm m_{ij}$, where $\{i,j\}$ is an edge of T. In particular, $\det B$ is not divisible by p. $\square$

To derive a graphical characterization of *all* quasiperfect Coxeter groups, we associate to each Coxeter graph Γ an *even* Coxeter group $E(\Gamma)$.

Let C be an indexing set for the connected components of $\Gamma^{(2)}$. For $c \in C$, let $I_c \subseteq I$ denote the vertex set of the corresponding connected component of $\Gamma^{(2)}$. For each pair $c, d \in C$, define

$$e_{cd} := \gcd\{m_{ij} \colon i \in I_c,\ j \in I_d\} \in 2\mathbb{Z}. \tag{9}$$

Let $\Psi = \Psi(\Gamma)$ be the labeled graph on vertex set C with edge $\{c,d\}$ labeled e_{cd}. Then we say that Ψ is the *even Coxeter graph*, and $E = E(\Gamma) = W(\Psi)$ the *even Coxeter group*, associated to Γ. Note that, if E is presented on generators $T = \{t_c \colon c \in C\}$, then the map $s_i \mapsto t_c$, where $i \in I_c$, extends to an epimorphism $W \twoheadrightarrow E$.

For each $c \in C$, let Δ_c denote the subgraph of Γ on the vertex set I_c. We can now state our characterization of quasiperfect Coxeter groups.

Theorem 5.2. *Let Γ be a Coxeter graph, and $W = W(\Gamma)$. Let C, and Δ_c $(c \in C)$, be as defined above, and $\Psi = \Psi(\Gamma)$ the even Coxeter graph associated to Γ. Then W is quasiperfect if and only if the following hold:*

- (a) *for all $c \in C$, and all primes p, $(\Delta_c)^{(p)}$ is connected; and*
- (b) *every edge of Ψ is labeled 2.*

Proof. Suppose that W is quasiperfect. For each $c \in C$, the map $s_i \mapsto s_i$ $(i \in I_c)$ and $s_i \mapsto 1$ $(i \in I \setminus I_c)$ extends to an epimorphism $W \twoheadrightarrow W(\Delta_c)$. Hence, in view of Proposition 5.1, condition (a) holds. Similarly, considering the natural epimorphism $W \twoheadrightarrow E$, Corollary 4.5 shows that condition (b) holds.

Conversely, assume that conditions (a) and (b) hold for $W = W(\Gamma)$. Fix $c \in C$, $i_c \in I_c$, and put $t_c := s_{i_c}$. Condition (a) and Proposition 5.1 state that $U := W(\Delta_c)$ is quasiperfect. Hence, for each $n \geqslant 0$, we have $U^{(n)} s_i = U^{(n)} t_c$ for all $i \in I_c$. It follows that, for all $c \in C$, $i \in I_c$, and $n \geqslant 0$, we have $W^{(n)} s_i = W^{(n)} t_c$. By Lemma 2.1, the map $W \to E$ induces an epimorphism $W/W^{(n)} \twoheadrightarrow E/E^{(n)}$ for each $n \geqslant 0$. However, each such map is injective, since E is the quotient of W by the normal closure of $\{s_i s_j : i, j \in I_c \text{ for some } c \in C\} \subseteq W^{(n)}$. It now follows from condition (b) that W is quasiperfect. $\square$

Theorem 5.2 is a graphical characterization of quasiperfect Coxeter groups; the following immediate corollary provides a more structural interpretation.

Corollary 5.3. *Let Γ be a Coxeter graph, and $W = W(\Gamma)$. Let C, and Δ_c $(c \in C)$, be as defined above, and $E = E(\Gamma)$ the even Coxeter group associated to Γ. Then W is quasiperfect if and only if the following hold:*

- (a) *for all $c \in C$, $W(\Delta_c)$ is quasiperfect; and*
- (b) *E is an elementary abelian 2-group.*

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