Solutions to Complex Arithmetic Practice Problems

Rectangular to polar problem set:
To convert to polar form, must find magnitude and phase.

1. mag. = \sqrt{(18)^2 + (15)^2} = 23.4 \quad \text{phase} = \tan^{-1} \left( \frac{15}{18} \right) = 0.695 \text{ rad}
   \rightarrow 18 + j15 = 23.4e^{j0.695} = 23.4\angle39.8^\circ

2. mag. = \sqrt{(-3)^2 + (-21)^2} = 21.2 \quad \text{phase} = \tan^{-1} \left( \frac{-21}{-3} \right) = 1.43 + \pi = 4.57 \text{ rad}
   (Note that some calculators don’t place the result of the inverse tangent function in the correct quadrant when the real part is negative.)
   \rightarrow -3 - j21 = 21.2e^{j4.57} = 21.2\angle-98^\circ

3. mag. = \sqrt{(-3)^2 + (21)^2} = 21.2 \quad \text{phase} = \tan^{-1} \left( \frac{21}{-3} \right) = -1.43 + \pi = 1.71 \text{ rad}
   \rightarrow -3 + j21 = 21.2e^{j1.71} = 21.2\angle98^\circ

4. mag. = \sqrt{a^2 + b^2} \quad \text{phase} = \tan^{-1} \left( \frac{b}{a} \right)
   \rightarrow a + jb = \sqrt{a^2 + b^2}e^{\tan^{-1}(b/a)} = \sqrt{a^2 + b^2}\angle\tan^{-1}(b/a)

5. mag. = \sqrt{\cos^2\omega t + \sin^2\omega t} = 1 \quad \text{phase} = \tan^{-1} \left( \frac{\sin\omega t}{\cos\omega t} \right) = \tan^{-1} (\tan \omega t) = \omega t
   \rightarrow \cos \omega t + j \sin \omega t = e^{j\omega t} (Euler’s formula!)

6. mag. = \sqrt{\sin^2\omega t + \cos^2\omega t} = 1
   \text{phase} = \tan^{-1} \left( \frac{\cos\omega t}{\sin\omega t} \right) = \tan^{-1} \left[ \frac{\cos(-\omega t)}{\sin(-\omega t)} \right] = \tan^{-1} \left[ \frac{\sin(-\omega t + \pi/2)}{\cos(-\omega t - \pi/2)} \right] = \tan^{-1} \left[ \frac{\sin(\pi/2 - \omega t)}{\cos(\pi/2 - \omega t)} \right]
   = \tan^{-1} \left[ \tan(\pi/2 - \omega t) \right] = \pi/2 - \omega t
   \rightarrow \sin \omega t + j \cos \omega t = e^{j(\pi/2-\omega t)} = 1\angle \left( 90^\circ - \omega t \frac{360^\circ}{2\pi} \right) (this is correct phasor form)
   this is also equal to \( e^{j(\pi/2-\omega t)} = e^{j\pi/2}e^{-j\omega t} = je^{-j\omega t} \)
7. mag. = \sqrt{9^2 \cos^2 \omega t + 3^2 \sin^2 \omega t} = \sqrt{81 \left(1 - \sin^2 \omega t\right) + 9 \sin^2 \omega t} = \sqrt{81 - 72 \sin^2 \omega t}

(magnitude varies between 3 and 9, depending on the time \( t \))

phase = \tan^{-1} \left(\frac{3 \sin \omega t}{9 \cos \omega t}\right) = \tan^{-1} \left(\frac{1}{3} \tan \omega t\right) \neq \frac{1}{3} \omega t

(phase angle does not vary linearly with time as in the case of \( e^{j\omega t} \))

Polar to rectangular problem set:

1. \( 8e^{-j0.12} = 8 [\cos(-0.12) + j \sin(-0.12)] = 8 [\cos(0.12) - j \sin(0.12)] = 7.94 - j0.096 \)

2. \( 14 \angle 132^\circ = 14 [\cos(132^\circ) + j \sin(132^\circ)] = -9.37 + j10.4 \)

3. \( -6 \angle -85^\circ = -6 [\cos(-85^\circ) + j \sin(-85^\circ)] = -6 [\cos(85^\circ) - j \sin(85^\circ)] = -0.523 + j5.98 \)

4. \( Ae^{j\omega t} = A (\cos \omega t + j \sin \omega t) = A \cos \omega t + jA \sin \omega t \)

5. \( Be^{-j(\omega t+\pi/6)} = B [\cos(-\omega t - \pi/6) + j \sin(-\omega t - \pi/6)] \)

\( \quad = B [\cos(\omega t + \pi/6) - j \sin(\omega t + \pi/6)] = B \cos(\omega t + \pi/6) - jB \sin(\omega t + \pi/6) \)

6. \( 15e^{j(\omega t-\pi/2)} = 15e^{-j\pi/2}e^{j\omega t} = -j15e^{j\omega t} = -j15(\cos \omega t + j \sin \omega t) \)

\( \quad = -j15 \cos \omega t - j^215 \sin \omega t = 15 \sin \omega t - j15 \cos \omega t \)

7. \( e^{-j3\pi/2} = \cos \left(\frac{-3\pi}{2}\right) + j \sin \left(\frac{-3\pi}{2}\right) = 0 + j = j \)

Express in proper polar form:

Proper polar form requires a complex number to be expressed in terms of a positive magnitude and a phase angle (phase can be expressed either in exponential form or in angular form; that is, either “mag \( e^{j\phase} \)” or “mag \( \angle \phase \)”).

1. \( -10e^{-j3\pi/2} = 10(-1)e^{-j3\pi/2} = 10e^{j\pi}e^{-j3\pi/2} = 10e^{j(\pi-3\pi/2)} = 10e^{-j\pi/2} \)

or \( 10e^{-j\pi}e^{-j3\pi/2} = 10e^{j(-\pi-3\pi/2)} = 10e^{-j5\pi/2} = 10e^{-j2\pi}e^{-j\pi/2} = 10e^{-j\pi/2} \)

The usual practice is to give phase values in the range \(-\pi\) to \(\pi\).

Note that \( e^{j\pi} = \cos(\pi) + j \sin(\pi) = -1 + j0 = -1 \) and that
\[ e^{-j\pi} = \cos(-\pi) + j\sin(-\pi) = \cos(\pi) - j\sin(\pi) = -1 \quad (\pi \text{ rad} = 180^\circ). \]
Also note that \( e^{j2\pi} = e^{-j2\pi} = 1. \)

2. \( Be^{-j\omega t - \alpha t} = Be^{-j\omega t}e^{-\alpha t} = Be^{-\alpha t}e^{-j\omega t} \)

Note that \(-j\omega t - \alpha t\) does not represent a phase angle because it contains a real part \(-\alpha t\).

The magnitude in this case \( (Be^{-\alpha t}) \) is time-varying, which is okay.

Find real and imaginary parts:

1. \( \text{Re}\{8e^{-j0.12}\} = \text{Re}\{7.94 - j0.096\} = 7.94 \)

   The result \( 7.94 - j0.096 \) is from the earlier polar-to-rectangular problem set.

   \( \text{Im}\{8e^{-j0.12}\} = \text{Im}\{7.94 - j0.096\} = -0.096 \)

2. \( \text{Re}\{14\angle132^\circ\} = \text{Re}\{-9.37 + j10.4\} = -9.37 \)

   The result \(-9.37 + j10.4\) is from the earlier polar-to-rectangular problem set.

   \( \text{Im}\{14\angle132^\circ\} = \text{Im}\{-9.37 + j10.4\} = 10.4 \)

3. \( \text{Re}\{2\} = \text{Re}\{2 + j0\} = 2 \)

   \( \text{Im}\{2\} = 0 \)

4. \( \text{Re}\{j15\} = \text{Re}\{0 + j15\} = 0 \)

   \( \text{Im}\{j15\} = 15 \)

5. \( \text{Re}\{\sin \omega t\} = \text{Re}\{\sin \omega t + j\} = \sin \omega t \)

   \( \text{Im}\{\sin \omega t\} = 0 \)

6. \( \text{Re}\{j \cos \omega t\} = \text{Re}\{0 + j \cos \omega t\} = 0 \)

   \( \text{Im}\{j \cos \omega t\} = \cos \omega t \)

7. \( \text{Re}\{x^2 + y^2 + j2xy\} = x^2 + y^2 \)

   \( \text{Im}\{x^2 + y^2 + j2xy\} = 2xy \)

Express in proper rectangular form:

1. \( j5(-3 + j20) = (j5)(-3) + (j5)(j20) = -j15 - 100 = -100 - j15 \)

   Proper form is to put the real part on the left and the imaginary part on the right.
2. \(-16(j2)^2 - j8(j3)^2 = -16(j^2)(2^2) - j8(j^2)(3^2) = -16(-1)(4) - j8(-1)(9) = 64 + j72\)

3. \((j\beta + \alpha)^2 = (j\beta + \alpha)(j\beta + \alpha) = (j\beta)(j\beta) + 2(j\beta\alpha) + \alpha^2 = -\beta^2 + j2\alpha\beta + \alpha^2 = \alpha^2 - \beta^2 + j2\alpha\beta\)

   The real part is \(\alpha^2 - \beta^2\), and the imaginary part is \(2\alpha\beta\).

Find magnitudes and phases:

1. \(e^{-j\pi/2} = 1e^{-j\pi/2}\)

   By inspection, the magnitude is \(1\) \(|e^{-j\pi/2}| = 1\),

   and the phase is \(-\pi/2\) (or \(3\pi/2\), since \(e^{-j\pi/2} = e^{j3\pi/2}\)).

2. \(5e^{-j\pi}\)

   By inspection, the magnitude is \(5\) \(|5e^{-j\pi}| = 5\),

   and the phase is \(-\pi\) (or \(\pi\)).

   Also, note that \(5e^{-j\pi} = 5e^{j\pi} = 5(-1) = -5\).

3. \(-9e^{-j\pi/3} = 9(-1)e^{-j\pi/3} = 9e^{j\pi}e^{-j\pi/3} = 9e^{j(\pi-\pi/3)} = 9e^{j2\pi/3}\)

   or \(-9e^{-j\pi/3} = 9e^{-j\pi}e^{-j\pi/3} = 9e^{j(-\pi-\pi/3)} = 9e^{-j4\pi/3}\)

   \(-9e^{-j\pi/3}\) = 9 and

   \(\text{phase} \{-9e^{-j\pi/3}\} = 2\pi/3\) or \(-4\pi/3\)

   Note that the angle \(2\pi/3\) (120°) is equivalent to the angle \(-4\pi/3\) (-240°).

4. \(|2 + j3| = \sqrt{(2)^2 + (3)^2} = 3.61\)

   \(\text{phase}\{2 + j3\} = \tan^{-1}(3/2) = 0.983\) (56.3°)

   Thus, \(2 + j3 = 3.61e^{j0.983} = 3.61\angle 56.3^\circ\)

5. \(|-j8| = \sqrt{(0)^2 + (-8)^2} = 8\)

   \(\text{phase}\{-j8\} = \tan^{-1}(-8/0) = \tan^{-1}(-\infty) = -\pi/2\) (-90°)

6. \(|jFe^{j0.83\pi}| = |j||F||e^{j0.83\pi}|\)

   (The magnitude of a product is equal to the product of the magnitudes.)

   Since \(|j| = 1\) and \(|e^{j0.83\pi}| = 1\), then

   \(|jFe^{j0.83\pi}| = |F| = F|
To find the phase, put the expression into proper polar form (using $j = e^{j\pi/2}$):

$$jF e^{j0.83\pi} = F e^{j\pi/2} e^{j0.83\pi} = F e^{j(\pi/2 + 0.83\pi)} = F e^{j1.33\pi}$$

→ phase\{jF e^{j0.83\pi}\} = 1.33\pi (239.4°) by inspection

The phase can also be expressed as $1.33\pi - 2\pi = -0.67\pi (-120.6°)$.

7. The solution is very similar to that of Problem 6.

$$|jF e^{jg}| = |j||F||e^{jg}| = (1)(F)(1) = F$$

To find the phase, put the expression into proper polar form:

$$jF e^{jg} = F e^{j\pi/2} e^{jg} = F e^{j(\pi/2 + g)}$$

→ phase\{jF e^{jg}\} = \pi/2 + g by inspection

8. $|-15| = 15$

Put the expression into proper polar form:

$$-15 = 15(-1) = 15e^{\pm j\pi}$$

→ phase\{-15\} = $\pi$ or $-\pi$ ($\pm 180°$)

9. $j = e^{j\pi/2}$, so $|j| = 1$

The phase is $\pi/2$ by inspection.

Find complex conjugates:

To find the complex conjugate of a complex number or expression, simply replace every occurrence of $j$ with $-j$. The conjugate of a sum is equal to the sum of the conjugates, and the conjugate of a product is equal to the product of the conjugates.

1. $j^* = -j$

2. $(2 + j7 - Fe^{jg})^* = 2 - j7 - Fe^{-jg}$

3. $(3 \times 10^{-6})^* = 3 \times 10^{-6}$

   Real numbers are unaffected by complex conjugation.

4. $[Re \{5 + j15\}]^* = (5)^* = 5$

5. $[Im \{5 + j15\}]^* = (15)^* = 15$

6. $[-j832e^{-0.3z}e^{(7-j18)t}]^* = j832e^{-0.3z}e^{(7+j18)t}$

7. $[8e^{-j0.12}(0.3 + j6.1)]^* = 8e^{j0.12}(0.3 - j6.1)$