# ENGR 695 Advanced Topics in Engineering Mathematics Spring 2021

**Homework Assignment #3 – due via Moodle at 11:59 pm on Friday, Oct. 6, 2023**
**[Prob. 3b revised and additional text in boldface added 10/4/23]**

***Instructions, notes, and hints*:**

You may make reasonable assumptions and approximations in order to compensate for missing information, if any. Provide the details of all solutions, including important intermediate steps. You will not receive credit if you do not show your work.

Unless otherwise specified, you may use *Matlab* or other mathematic software (or a calculator) to check your work.

It is your responsibility to review the solutions when they are posted and to understand and rectify any conceptual errors that you might have. You may contact me at any time for assistance.

The first set of problems will be graded and the rest will not be graded. Only the graded problems must be submitted by the deadline above; do not submit the ungraded problems.

***Graded Problems*:**

**1.** An orthogonal matrix is one in which the column vectors that form the matrix are orthogonal.

a. For an orthogonal matrix $A$, show that the result of the operation $A^TA$ must produce a diagonal matrix.
b. Given the result of part a, if $A$ is an orthonormal matrix, what is distinctive about the result of the operation $A^TA$? Use this result to prove that $A^{-1} = A^T$ for orthonormal matrices (a one or two-line proof).

**2.** Find an appropriate set of elements to fill the blank first column of the following matrix to make the matrix orthonormal (i.e., fill the three boxes with appropriate values). Find the values using a deterministic approach, not by trial-and-error.

$$\begin{bmatrix} \square & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \square & 0 & \frac{1}{\sqrt{3}} \\ \square & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

(*continued on next page*)

**3.** Determine whether the following problems are initial value problems (IVPs), boundary value (BVPs), or neither. Briefly explain your answer for each part.

a. $y'' + a^2 y = 0$ with $y(0) = 1$ and $y(1) = 0$

b. **[boldface text corrected 10/4/23]** $y'' + a^2 y = 0$ with $y(0) = 0$ and $\boldsymbol{y'(0) = 1}$

**4.** Find solutions to the Fourier equation $y'' + a^2 y = 0$ for the following sets of boundary conditions. **After you complete the solutions, think about the implications of the differences between them.**

a. $y(0) = 1$ and $y(1) = 0$

b. $y(0) = 0$ and $y(1) = 0$

**5.** Find solutions to the modified Fourier equation $y'' - a^2 y = 0$ for the following sets of boundary conditions. **After you complete the solutions, think about the implications of the differences between them.**

a. $y(0) = 1$ and $y(1) = 0$

b. $y(0) = 0$ and $y(1) = 0$

**6.** Find the eigenvalues and eigenfunctions of the BVP given below. **Consider the conditions $\boldsymbol{\lambda < 0}$, $\boldsymbol{\lambda = 0}$, and $\boldsymbol{\lambda > 0}$.**

$$y'' + \lambda y = 0 \text{ with } y(0) = 1 \text{ and } y'(\pi/2) = 0$$

**[This is just a comment. It is not meant to suggest a solution strategy for this problem:]** Note the similarity between this problem and the matrix eigensystem defined by $A\mathbf{x} = \lambda\mathbf{x}$. Recall that the second finite differences of a set of discrete values contained in a vector $\mathbf{x}$ can be found by multiplying $\mathbf{x}$ by the matrix $D$ given below. If the dimensions (size) of the matrix expression $-D\mathbf{y} = \lambda\mathbf{y}$ approaches infinity, then the matrix expression becomes equivalent to the ODE given above. (The minus sign can be absorbed into $D$.)

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & \\ 0 & 1 & -2 & 1 & & \vdots \\ \vdots & & & \ddots & & \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}$$

**7.** Find the eigenvalues and eigenfunctions of the BVP given below. **Consider the conditions $\boldsymbol{\lambda < -1}$, $\boldsymbol{\lambda = -1}$, and $\boldsymbol{\lambda > -1}$.**

$$y'' + (\lambda + 1) y = 0 \text{ with } y'(0) = 1 \text{ and } y'(1) = 0$$

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***Ungraded Problems*:**

The following problems will not be graded. However, you should attempt to solve them on your own and then check the solutions. Try not to give up too quickly if you struggle to solve them.

**1.** Show that an orthonormal matrix that multiplies a vector does not change the length of the vector. That is, show that $\mathbf{x}$ and $Q\mathbf{x}$ (where $Q$ is an orthonormal matrix) have the same length. Note that the length of a vector is given by $x = (\mathbf{x}^T\mathbf{x})^{1/2}$.

**2.** Show that an orthonormal matrix does not change the angle between two vectors (i.e., it does not change their relative orientation). To do this, show that $\theta_{xy} = \theta_{ab}$, where $\mathbf{a} = Q\mathbf{x}$ and $\mathbf{b} = Q\mathbf{y}$. Note that the angle between two vectors $\mathbf{x}$ and $\mathbf{y}$ is given by the expression

$$\cos\theta_{xy} = \frac{\mathbf{x}^T\mathbf{y}}{xy},$$

where $x$ and $y$ are the lengths of the vectors $\mathbf{x}$ and $\mathbf{y}$, respectively.

**3.** [adapted from Prob. 30 in Sec. 3.9 of Zill, 6th ed.] The temperature distribution $u(r)$ in a circular ring or annulus is determined from the BVP given by

$$r\frac{d^2u}{dr^2} + \frac{du}{dr} = 0 \text{ with } u(a) = u_0 \text{ and } u(\mathrm{b}) = u_1,$$

where $u_0$ and $u_1$ are constants. Show that

$$u(r) = \frac{u_0 \ln(r/b) - u_1 \ln(r/a)}{\ln(b/a)}$$