ENGR 695 Advanced Topics in Engineering Mathematics Fall 2023

Homework Assignment #5 – due via Moodle at 11:59 pm on Thursday, Nov. 2, 2023
[Graded Prob. 3 revised 11/2/23]

***Instructions, notes, and hints*:**

You may make reasonable assumptions and approximations in order to compensate for missing information, if any. Provide the details of all solutions, including important intermediate steps. You will not receive credit if you do not show your work.

Unless otherwise specified, you may use *Matlab* or other mathematic software (or a calculator) to check your work.

It is your responsibility to review the solutions to the graded and ungraded problems when they are posted and to understand and rectify any conceptual errors that you might have. You may contact me at any time for assistance.

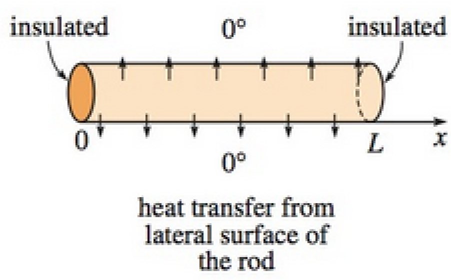
The first set of problems will be graded and the rest will not be graded. Only the graded problems must be submitted by the deadline above; do not submit the ungraded problem.

***Graded Problems*:**

1. [adapted from Prob. 5 in Sec. 13.3 of D. G. Zill, 6th ed.] Heat is lost from the lateral surface of a thin rod of length L into a surrounding medium that has a temperature of 0 K. If the linear law of heat transfer applies, then the heat equation has the form

$$k\frac{\partial^2 u}{\partial x^2} - hu = \frac{\partial u}{\partial t}, \qquad 0 \le x \le L \quad \text{and} \quad t \ge 0$$

where h is a constant. Find an expression for the temperature $u(x,t)$ if the initial temperature is $f(x)$ over the interval $[0, L]$ and the ends $x = 0$ and $x = L$ are insulated as shown in the figure below.



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2. Find a SOV solution to the 1-D heat equation given below left, where u = temperature, for a rod of length L = 2 m and insulated ends. The initial temperature distribution $f(x)$ along the rod is given below. The thermal diffusivity of the rod is k = 0.01 m^2/s. The solution should have only two nonzero coefficients in the set $\{A_n\}$. Use the hint below right to show that the coefficients for n = 3 and higher are all zero. You may use mathematical analysis software such as Matlab or Mathematica to perform the numerical integrations required to obtain the two coefficient values. You may also use your calculator if it has that capability.

$$k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t} \qquad f(x)=200\sin^2\left(\frac{\pi x}{L}\right) \qquad \textit{Hint}:\ \sin^2(x)=\frac{1}{2}-\frac{1}{2}\cos(2x)$$

3. **[text in boldface revised 11/1/23]** [adapted from Prob. 15 in Sec. 13.4 of D. G. Zill, 6th ed.] A string is stretched and secured on the x-axis at $x = 0$ and $x = \boldsymbol{\pi}$ for $t > 0$. If the transverse vibrations take place in a medium that imparts a resistance proportional to the instantaneous velocity, then the governing wave equation has the form

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}+2\beta\frac{\partial u}{\partial t}, \qquad 0<\beta<1, \quad t>0\,.$$

Find an expression for the displacement $u(x,t)$ if the string starts from rest from the initial displacement $f(x)$. A string that starts from rest has $g(x) = 0$.

***Ungraded Problem*:**

The following problem will not be graded. However, you should attempt to solve it on your own and then check the solution. Try not to give up too quickly if you struggle to solve it.

1. Recall that the homogeneous boundary value problem involving the one-dimensional heat equation and its boundary conditions and initial condition can be expressed as

$$k\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t}, \qquad 0\le x\le L \quad \text{and} \quad t\ge 0$$

$$\text{with } u(0,t)=0 \quad u(L,t)=0 \quad u(x,0)=f(x),$$

where the dependent variable u represents the heat or temperature. The function $f(x)$ specifies the heat or temperature distribution along the spatial interval $[0, L]$ at time $t = 0$. Applying the SOV method and assuming that $u(x, t) = X(x)T(t)$ yields an expression of the form

$$\frac{X''}{X}=\frac{1}{k}\frac{T'}{T}=-\lambda\,,$$

where $-\lambda$ is the separation constant.

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Ungraded Prob. 1: (continued)

The resulting set of ordinary differential equations (ODEs) is

$$X'' + \lambda X = 0 \qquad \text{and} \qquad T' + k\lambda T = 0,$$

and the resulting solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-kn^2\pi^2 t/L^2} \sin\left(\frac{n\pi}{L}x\right), \quad \text{where} \quad A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx.$$

Note that the thermal diffusivity k could have been associated with the ODE in $X(x)$ instead of the ODE in $T(t)$. That is, the application of the SOV method could have led to the alternative expression

$$k\frac{X''}{X} = \frac{T'}{T} = -\lambda.$$

Show that the solution for $u(x, t)$ given above is still obtained when the alternative expression is used as the starting point.