ENGR 695 Advanced Topics in Engineering Mathematics Fall 2023

Homework Assignment #7 – due via Moodle at 11:59 pm on Friday, Nov. 17

You may make reasonable assumptions and approximations to compensate for missing information, if any. Provide the details of all solutions, including important intermediate steps. You will not receive credit if you do not show your work. You may use *Matlab* or other software to check your work.

It is your responsibility to review the solutions to the graded and ungraded problems when they are posted and to understand and rectify any conceptual errors that you might have. You may contact me at any time for assistance.

The first set of problems will be graded and the rest will not be graded. Only the graded problems must be submitted by the deadline above; do not submit the ungraded problems.

***Graded Problems*:**

1. Derive an explicit finite difference update equation for the numerical solution of the two-dimensional heat equation problem given by

$$c\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\frac{\partial u}{\partial t}, \qquad 0\le x\le a, \quad 0\le y\le b, \quad \text{and} \quad t\ge 0\,,$$

where c is the thermal diffusivity of the material. The spatial locations within the solution space are defined by $x = (i - 1)\Delta x$ for $i = 1, 2, 3, \ldots, N_x$ and $y = (j - 1)\Delta x$ for $j = 1, 2, 3, \ldots, N_y$, where N_x and N_y are the total number of discrete points in the x and y directions, respectively. The time variable is indexed as $t = (n - 1)\Delta t$ for $n = 1, 2, 3, \ldots, N_t$, where N_t is the total number of time steps. Group like terms in the update equation so that the number of floating-point operations is minimized. Use the derivation at the beginning of Sec. 16.2 of the textbook (D. G. Zill, *Advanced Engineering Mathematics*, 6th ed.) as a guide, and use the following constants to simplify the update equation:

$$C_x=\frac{c\Delta t}{\Delta x^2} \quad \text{and} \quad C_y=\frac{c\Delta t}{\Delta y^2}\,.$$

2. Derive an explicit finite difference update equation for the following wave equation, which models a medium that offers resistance proportional to the instantaneous transverse velocity of the disturbance.

$$v^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}+2\beta\frac{\partial u}{\partial t}, \qquad 0\le x\le L, \quad 0\le \beta\le 1, \quad \text{and} \quad t\ge 0$$

The FD approximations must all be spatially and temporally centered. Use the spatial and temporal indices i and n, respectively, defined by $x = (i - 1)\Delta x$ for $i = 1, 2, 3, \ldots, N_x$, and $t = n\,\Delta t$ for $n = 0, 1, 2, \ldots, N_t$, where N_x and N_t are the numbers of locations and time steps, respectively. Group like terms in the update equation so that the number of floating-point operations is minimized. You do not have to find the special update equation for the $n = 0$ (corresponding to $t = 0$) case.

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3. Derive an explicit finite difference update equation for the numerical solution of the following PDE, which models a string initially at rest on the *x*-axis but that is allowed to fall under its own weight for $t > 0$.

$$a^2 \frac{\partial^2 u}{\partial x^2} - g = \frac{\partial^2 u}{\partial t^2}, \qquad 0 \le x \le L \quad \text{and} \quad t \ge 0$$

The dependent variable $u(x, t)$ is the vertical displacement of the string, and the constant g is the acceleration of gravity. The FD approximations must use spatially and temporally centered differences. Use space and time grids defined by $x = (i - 1)\Delta x$ for $i = 1, 2, 3, \ldots, N_x$, where N_x is the total number of spatial points, and $t = (n - 1)\Delta t$ for $n = 1, 2, 3, \ldots, N_t$, where N_t is the total number of time steps. Group like terms in the update equation to minimize the number of floating-point operations. You do not have to find the special update equation for the $t = 0$ case.

4. Refer to the lecture notes "Open Boundaries in the Finite Difference Solution of the 1-D Wave Equation." The special update equations derived in the notes that are applied at the boundaries of the solution space are:

$$u_{1,j+1} = \frac{2C^2}{1+C} u_{2,j} + 2(1-C) u_{1,j} - \frac{1-C}{1+C} u_{1,j-1}, \quad \text{for } i = 1$$

$$u_{N_x,j+1} = 2(1-C) u_{N_x,j} + \frac{2C^2}{1+C} u_{N_x-1,j} - \frac{1-C}{1+C} u_{N_x,j-1}, \quad \text{for } i = N_x, \qquad \text{where} \quad C = \frac{v_p \Delta t}{\Delta x}.$$

Consider the case when Δt is set to the maximum possible value that satisfies the Courant-Friedrichs-Lewy (CFL) condition for stability. Given that these two boundary conditions are based on the one-way wave equations, explain why the resulting update equations obtained at the limit of the CFL condition make sense.

***Ungraded Problem*:**

The following problem will not be graded. However, you should attempt to solve it on your own and then check the solution. Try not to give up too quickly if you struggle to solve it.

1. Show that the one-way wave equations

$$v \frac{\partial u_1}{\partial x} + \frac{\partial u_1}{\partial t} = 0 \quad \text{and} \quad v \frac{\partial u_2}{\partial x} - \frac{\partial u_2}{\partial t} = 0$$

have the respective solutions

$$u_1(x, t) = f(x - vt) \quad \text{and} \quad u_2(x, t) = f(x + vt),$$

where $f(x)$ is the initial condition. This is not a trick question; the solutions are almost trivial.