ENGR 695 Advanced Topics in Engineering Mathematics Fall 2023

Lab #7: The Separation Constant in the Separation of Variables Method

Introduction

In this lab session, you will examine an aspect of the separation of variables method that emerged during the lecture sessions but was deferred until later. You should also gain addition insight into why the solutions to homogeneous boundary value problems take the forms that they do. This lab exercise will not require any *Matlab* coding; your deliverable will be a handwritten (or word-processed, if you wish) response.

Procedure

Recall that the homogeneous boundary value problem involving the one-dimensional heat equation and its boundary conditions and initial condition can be expressed as

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad 0 \le x \le L \quad \text{and} \quad t \ge 0$$

$$\text{with } u(0,t) = 0 \quad u(L,t) = 0 \quad u(x,0) = f(x),$$

where the dependent variable u represents the heat or temperature. The function $f(x)$ specifies the heat or temperature distribution along the spatial interval $[0, L]$ at time $t = 0$. Applying the SOV method and assuming that $u(x, t) = X(x)T(t)$ yields an expression of the form

$$\frac{X''}{X} = \frac{1}{k}\frac{T'}{T} = -\lambda,$$

where $-\lambda$ is the separation constant. The minus sign is added to the separation constant because experience with these types of problems has shown that it simplifies the analysis. However, the minus sign is not mathematically necessary. The separation expression could have the form

$$\frac{X''}{X} = \frac{1}{k}\frac{T'}{T} = \lambda.$$

Show that the solution to the problem is still

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) e^{-kn^2\pi^2 t/L^2}, \quad \text{where} \quad A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

if the expression with an unnegated separation constant λ is used as the starting point.

Hint #1: The usual assumed solution to the ordinary differential equation (ODE) $X'' - \lambda X = 0$ with closed boundaries has the form

$$X(x) = c_1 \cosh\left(\sqrt{\lambda}\, x\right) + c_2 \sinh\left(\sqrt{\lambda}\, x\right).$$

If λ is positive, then applying the boundary conditions to the assumed form above results in the trivial solution. However, consider employing the identities

$$\cosh(x) = \cos(ix) \qquad \text{and} \qquad \sinh(x) = -i \sin(ix),$$

where i is the square root of -1, and then applying the boundary conditions. For your own edification, you should confirm that the alternative solution is in fact a solution to the ODE. That is, substitute the expression for $X(x)$ obtained after incorporating the identities above into the equation $X'' - \lambda X = 0$.

Hint #2: As you complete the solution with the alternative form above, you should find that the equation $X'' - \lambda X = 0$ with homogeneous boundary conditions can only have a nontrivial solution if λ is negative. In essence, the modified Fourier equation becomes a Fourier equation because the alternative solution "corrects" for the "wrong" algebraic sign of the separation constant.

Lab Work Submission and Scoring

After you complete the exercise, scan or convert your written work into PDF format, and upload the file using the submission link at the course Moodle site. If you must photograph your work, insert the photos into a single word-processed document and then convert the document to PDF format.

Your score will be assigned according to the homework scoring rubric posted on the Syllabus and Policies page at the course web site.

If you do not complete the exercises during the lab session, then you may submit your documentation as late as 11:59 pm on Friday, November 3. If the file is submitted after the deadline, a 5% score deduction will be applied for every 24 hours or portion thereof that the item is late (not including weekend days) unless extenuating circumstances apply. No credit will be given five or more days after the deadline.