ENGR 695 Advanced Topics in Engineering Mathematics Fall 2023

Lecture Outline for Friday, Oct. 13

1. Typos in HW #3 Problems 6 and 7. They were supposed to have homogeneous BCs. Solving the intended problems now is a good lead-in to the topic of Sturm-Liouville problems.

   a. intended form of Prob. 6:

   $$y'' + \lambda y = 0 \text{ with } \boldsymbol{y(0) = 0} \text{ and } y'(\pi/2) = 0$$

   b. $\lambda < 0$ case (use change of variables $\lambda = -\alpha^2$, where $\alpha > 0$); closed boundaries, so

   $$y(x) = c_1 \cosh(\alpha x) + c_2 \sinh(\alpha x).$$

   Apply first BC:

   $$y(0) = c_1 \cosh(0) + c_2 \sinh(0) \quad \rightarrow \quad 0 = c_1(1) + c_2(0) \quad \rightarrow \quad c_1 = 0.$$

   First derivative (with c_1 now equal to 0):

   $$y'(x) = c_2 \alpha \cosh(\alpha x).$$

   Apply second BC:

   $$y'(\pi/2) = 0 = c_2 \alpha \cosh(\alpha \pi/2) \quad \rightarrow \quad c_2 = 0 \text{ (valid for all values of } \alpha)$$

   → No nontrivial solutions.

   c. $\lambda = 0$ case:

   $$y(x) = c_1 + c_2 x \quad \text{and} \quad y'(x) = c_2$$

   Apply BCs:

   $$y(0) = 0 = c_1 + c_2(0) \quad \rightarrow \quad c_1 = 0 \quad \text{and} \quad y'(\pi/2) = 0 = c_2$$

   → No nontrivial solutions.

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d. $\lambda > 0$ case (use change of variables $\lambda = \alpha^2$, where $\alpha > 0$); closed boundaries, so

$$y(x) = c_1 \cos(\alpha x) + c_2 \sin(\alpha x) \quad \text{and} \quad y'(x) = -c_1\alpha \sin(\alpha x) + c_2\alpha \cos(\alpha x).$$

Apply first BC:

$$y(0) = 0 = c_1 \cos(0) + c_2 \sin(0) \quad \rightarrow \quad 0 = c_1(1) + c_2(0) \quad \rightarrow \quad c_1 = 0.$$

Apply second BC:

$$y'(\pi/2) = 0 = c_2\alpha \cos(\alpha\,\pi/2) \quad \rightarrow \quad \alpha_n = n, \text{ for odd } n$$

→ Eigenvalue problem.

2. Important general form of ODE: the regular Sturm-Liouville equation

$$\frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] + q(x)y + \lambda p(x)y = 0$$

a. some textbooks use $p(x)$ for $r(x)$ and $w(x)$ for $p(x)$
b. second-order with variable coefficients
c. form above is called "self-adjoint" form
d. λ is a parameter in the problem (eigenvalue)
e. $r(x)$, $p(x) > 0$ on interval of solution
f. homogeneous boundary conditions defined over interval (a, b):

$$A_1 y(a) + B_1 y'(a) = 0$$
$$A_2 y(b) + B_2 y'(b) = 0$$

g. importance for our purposes:
   i. guarantees orthogonality, completeness, representation
   ii. function $p(x)$ defines kernel for inner product

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3. The "Sturm-Liouville Insurance Policy" (SLIP)

a. There are non-trivial solutions for specific values of the parameter λ (eigenvalues).
b. There is an infinity of eigenvalues.
c. There is a smallest but not a largest eigenvalue.
d. The eigenvalues are real and distinct ($\lambda_1 < \lambda_2 < \lambda_3 < \ldots$ such that $\lambda_n \to \infty$ as $n \to \infty$).
e. For each eigenvalue there is a single solution (eigenfunction) $y_n(x)$
f. The eigenfunctions corresponding to two different eigenvalues are orthogonal with respect to the weight function $p(x)$. That is,

$$\langle y_m, y_n \rangle_{p(x)} = \int_a^b y_m(x) y_n(x) p(x)\, dx = \begin{cases} 0, & m \neq n \\ C_m, & m = n \end{cases}$$

g. The set of solutions to an S-L problem are complete in that the set forms a basis for the space of square-integrable functions on the interval [a, b].

$$f(x) = \sum_{n=1}^{\infty} a_n y_n(x) \quad \text{with} \quad a_n = \frac{\langle f(x), y_n(x) \rangle}{\langle y_n(x), y_n(x) \rangle}$$

Proof:

Multiply both sides by $y_m(x)$ and evaluate the inner product. Don't forget the "weight" function $p(x)$:

$$\int_a^b f(x) y_m(x) p(x)\, dx = \sum_{n=1}^{\infty} a_n \int_a^b y_m(x) y_n(x) p(x)\, dx$$

Because of the SLIP, we know that the eigenfunctions $y_n(x)$ are orthogonal. Thus,

$$\int_a^b f(x) y_m(x) p(x)\, dx = a_m \int_a^b y_m(x) y_m(x) p(x)\, dx$$

Expressed in inner product notation,

$$\langle f(x), y_m(x) \rangle = a_m \langle y_m(x), y_m(x) \rangle \quad \to \quad a_m = \frac{\langle f(x), y_m(x) \rangle}{\langle y_m(x), y_m(x) \rangle}$$

The expression for a_n above follows after changing the index from m to n.