ENGR 695 Advanced Topics in Engineering Mathematics Fall 2023

Lecture Outline for Friday, Sept. 15

1. Eigensystem insights and expectations

   a. Simple eigensystem has N distinct eigenvalues, N linearly independent eigenvectors
   b. $A\mathbf{x}_i = \lambda\mathbf{x}_i \quad \rightarrow \quad AX = X\Lambda$
   c. Eigenvalues of upper and lower-triangular matrices are the diagonal values
   d. Eigenvalues of diagonal matrices are the diagonal values

2. Example: Compute eigenvalues and eigenvectors of A

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

3. Some basic theorems

   a. If A is square with real entries, then any complex eigenvalues/eigenvectors come in conjugate pairs.
   b. If A is square, then 0 is an eigenvalue only iff A is singular.
   c. $\det(A) = \lambda_1\lambda_2\lambda_3\ldots\lambda_N$
   d. If A is nonsingular and λ is an eigenvalue, then $1/\lambda$ is an eigenvalue of A^{-1}; both eigenvalues have the same corresponding eigenvectors.
   e. Eigenvalues of upper/lower-triangular and diagonal matrices are the diagonal values.

4. Important special cases

   a. Symmetric matrices ($A^T = A$) behave well
      i. Eigenvalues are real; all eigenvectors are linearly independent (LI)
      ii. Distinct eigenvalues → orthogonal eigenvectors (also LI)
      iii. Repeated eigenvalues → LI eigenvectors but might not be orthogonal
      iv. Linearly independent ≠ orthogonal
      v. Symmetric matrices can be singular and therefore have at least one zero eigenvalue; even so, all eigenvectors are LI.
   b. Orthogonal matrices ($A^{-1} = A^T$, which implies that $A^T A = I$)
      i. A is orthogonal iff its columns form an orthonormal set (orthonormal is orthogonal with each vector having a length of 1; i.e., $|\mathbf{x}| = \mathbf{x}^T\mathbf{x} = 1$)
      ii. Orthogonal matrices are not usually symmetric (The only orthogonal *and* symmetric matrix is I because a matrix that is both satisfies $A^{-1} = A^T = A$.)
      iii. Example: Check that $A^{-1} = A^T$ and that each column is normalized

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$$A = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

5. Where we are heading: *LU* and *QR* factorizations and the SVD