ENGR 695 Advanced Topics in Engineering Mathematics Fall 2023

Lecture Outline for Friday, Nov. 17

1. FD solutions of wave equation: computational considerations

    a. Accuracy generally improves as spatial step size Δx and/or time step size Δt is decreased, although not always (e.g., CFL condition is most accurate setting for Δt in explicit FD solution of wave equation)
    b. Grid dispersion: artificial change in velocity (usually frequency dependent or pulse rise/fall time dependent) due to discretization in space; reduce via small spatial step sizes
    c. Grid dissipation: artificial attenuation (usually frequency dependent or pulse rise/fall time dependent) due to discretization in space; reduce via small spatial step sizes
    d. Finite precision of computer representation of numbers becomes a problem for very small step sizes

2. FD solutions in non-Cartesian coordinate systems

    a. Challenging due to more complicated expressions and variable step sizes
    b. Often better to use Cartesian system and then apply a staircase approximation to irregular boundaries or interfaces between materials

3. Multiple materials in solution space:

    a. Additional interior boundary conditions might be necessary at interfaces
    b. Some FD methods inherently account for interior boundaries

4. Open boundaries in FD solution of wave equation

    a. One-way wave equation

$$v^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \quad \rightarrow \quad v^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0 \quad \rightarrow \quad \left(v^2 \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2} \right) u = 0 \quad \rightarrow \quad \left(v \frac{\partial}{\partial x} - \frac{\partial}{\partial t} \right)\left(v \frac{\partial}{\partial x} + \frac{\partial}{\partial t} \right) u = 0$$

    b. First factor describes a leftward traveling wave (in $-x$ direction), and second factor describes a rightward traveling wave (in $+x$ direction)
    c. Apply one of the two operators in parentheses at each boundary, which yields an open boundary condition (example: $x = a$ case, where waves should propagate out of space in the $-x$ direction):

$$\left(v \frac{\partial}{\partial x} - \frac{\partial}{\partial t} \right) u = 0 \quad \rightarrow \quad \left. \frac{\partial u}{\partial x} \right|_{x=a} = \frac{1}{v} \left. \frac{\partial u}{\partial t} \right|_{x=a}$$

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d. FD form of one-way wave equation at boundary (centered at $x = a$ or $i = 1$)

$$\frac{u_{2,j} - u_{0,j}}{2\Delta x} = \frac{1}{v}\left(\frac{u_{1,j+1} - u_{1,j-1}}{2\Delta t}\right)$$

leads to, after solving for $u_{0,j}$, special update equation used only at the $x = a$ (left-most) boundary

$$u_{1,j+1} = \frac{2C^2}{1+C} u_{2,j} + 2(1-C)u_{1,j} - \frac{1-C}{1+C} C u_{1,j-1},$$

$$\text{where} \quad C = \frac{v\Delta t}{\Delta x}$$

e. Similar update equation for use only at the $x = b$ (right-most) boundary. Start with

$$\left.\frac{\partial u}{\partial x}\right|_{x=b} = -\frac{1}{v_p} \left.\frac{\partial u}{\partial t}\right|_{x=b}$$

to obtain

$$u_{N_x, j+1} = 2(1-C)u_{N_x, j} + \frac{2C^2}{1+C} u_{N_x-1, j} - \frac{1-C}{1+C} u_{N_x, j-1}$$

f. Special case for first time step, $j = 0$. Start with initial condition applied at $x = a$:

$$g(a) = \left.\frac{\partial u}{\partial t}\right|_{t=0} \approx \frac{u(a, 0+\Delta t) - u(a, 0-\Delta t)}{2\Delta t} = \frac{u_{1,1} - u_{1,-1}}{2\Delta t},$$

which leads to

$$u_{1,1} = C^2 u_{2,0} + (1-C^2)u_{1,0} + (1-C)\Delta t\, g(a)$$

g. Similar result for j = 0 at $x = b$, the right-most boundary:

$$u_{N_x,1} = (1-C^2)u_{N_x,0} + C^2 u_{N_x-1,0} + (1-C)\Delta t\, g(b)$$

5. Next: Finite difference solution of the heat equation with Neumann BCs

$$c\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad \text{with} \quad \left.\frac{\partial u}{\partial x}\right|_{x=a} = u_{xa} \quad \text{and} \quad \left.\frac{\partial u}{\partial x}\right|_{x=b} = u_{xb},$$

where u_{xa} and u_{xb} are usually constants but could be time varying