ENGR 695 Advanced Topics in Engineering Mathematics Fall 2023

Lecture Outline for Monday, Sept. 18

1. Review of properties of symmetric real matrices

    a. $A^T = A$
    b. All eigenvalues are real; all eigenvectors are LI
    c. Distinct eigenvalues → orthogonal eigenvectors (also LI)
    d. Repeated eigenvalues → LI eigenvectors but might not be orthogonal
    e. Singular symmetric matrices have at least one zero eigenvalue; even so, all eigenvectors are LI.

2. Orthogonal matrices ($A^{-1} = A^T$, which implies that $A^T A = I$)

    a. A is orthogonal iff its columns form an orthonormal set (orthonormal is orthogonal with each vector having a length of 1; i.e., $|\mathbf{x}| = \mathbf{x}^T\mathbf{x} = 1$)
    b. Orthogonal matrices are not usually symmetric (The only orthogonal *and* symmetric matrix is I because a matrix that has both properties must satisfy $A^{-1} = A^T = A$.)
    c. Important application: Various kinds of diagonalizations, which can improve the efficiency of difficult computations and reveal skewness of basis vectors
    d. Example: Check that $A^{-1} = A^T$ and that the columns form an orthonormal set of vectors.

$$A = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

3. Diagonalization

    a. Express $N \times N$ matrix A as $A = PDP^{-1}$, where D is a diagonal matrix
    b. A does not have to be symmetric or orthogonal to be diagonalizable.
    c. Theorem: An $N \times N$ matrix A is diagonalizable iff A has N LI eigenvectors.
    d. Theorem: If $N \times N$ matrix A has N distinct eigenvalues then it is diagonalizable (but it might be diagonalizable even if the eigenvalues are not distinct, that is, if some are repeated).
    e. Theorem: An $N \times N$ matrix A can be orthogonally diagonalized iff A is symmetric. (*Orthogonal diagonalization* means that P is orthogonal.)
    f. Many ways to diagonalize a matrix.

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g. One important example: Given an $N \times N$ matrix A with LI eigenvectors, since $A\mathbf{x}_1 = \lambda_1\mathbf{x}_1$, $A\mathbf{x}_1 = \lambda_1\mathbf{x}_1$, etc., then, using the definitions

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_N \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_N \\ \downarrow & \downarrow & & \downarrow \end{bmatrix},$$

where X is formed by making its columns equal to the eigenvectors, we obtain $AX = X\Lambda \rightarrow A = X\Lambda X^{-1}$. Thus, an $N \times N$ matrix A with LI eigenvectors can be diagonalized into its eigenvalues and eigenvectors.

h. Example: Attempt to diagonalize the following matrix:

$$A = \begin{bmatrix} -5 & 9 \\ -6 & 10 \end{bmatrix}$$