ENGR 695 Advanced Topics in Engineering Mathematics Fall 2023

Lecture Outline for Wednesday, Nov. 1

1. Wave equation example

$$v^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad 0 \le x \le L, \quad t \ge 0$$

   a. Vibrations of string: $v = \sqrt{\dfrac{T}{\rho}}$, where T = tension in string, ρ = mass per unit length

   b. Typical BCs and ICs (two ICs because time problem is second order)

$$u(0,t) = 0, \quad u(L,t) = 0, \quad u(x,0) = f(x), \quad \text{and} \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$$

   c. Apply SOV method with $u(x, t) = X(x)T(t)$ to obtain

$$v^2 X''T = XT'' \quad \rightarrow \quad \frac{X''}{X} = \frac{T''}{v^2 T} = -\lambda \quad \rightarrow \quad X'' + \lambda X = 0 \quad \text{and} \quad T'' + v^2 \lambda T = 0$$

   d. The BCs $u(0, t) = 0 \rightarrow X(x) = 0$ and $u(L, t) = 0 \rightarrow X(L) = 0$ lead to the orthogonal spatial eigenfunctions

$$X_n(x) = c_2 \sin\left(\frac{n\pi x}{L}\right) \quad \text{with} \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2 \quad \text{for} \quad n = 1, 2, 3, \ldots$$

   e. The T problem is an initial value problem involving the Fourier equation. The possible solution forms are

$$T_n(t) = c_4 \cos\left(v\sqrt{\lambda_n} t\right) + c_5 \sin\left(v\sqrt{\lambda_n} t\right) = c_4 \cos\left(\frac{n\pi v t}{L}\right) + c_5 \sin\left(\frac{n\pi v t}{L}\right)$$
$$\text{or} \quad T_n(t) = c_4 e^{iv\sqrt{\lambda_n} t} + c_5 e^{-iv\sqrt{\lambda_n} t},$$

   but the one that uses trigonometric functions is more convenient since it involves only real quantities.

   f. Each eigensolution to the full PDE with $A_n = c_2 c_4$ and $B_n = c_2 c_5$ has the form:

$$u_n(x,t) = X_n(x) T_n(t) = \left[A_n \cos\left(\frac{n\pi v t}{L}\right) + B_n \sin\left(\frac{n\pi v t}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right),$$

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g. Each eigensolution u_n is a solution, but a linear combination of eigensolutions is also a solution. Full solution to the PDE for the general case:

$$u(x,t)=\sum_{n=1}^{\infty}u_n(x,t)=\sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{n\pi vt}{L}\right)+B_n\sin\left(\frac{n\pi vt}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)$$

First time derivative (needed when second initial condition is applied):

$$\frac{\partial u}{\partial t}=\sum_{n=1}^{\infty}\left[-A_n\frac{n\pi v}{L}\sin\left(\frac{n\pi vt}{L}\right)+B_n\frac{n\pi v}{L}\cos\left(\frac{n\pi vt}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)$$

h. Determination of coefficients in summation. Two sets of coefficients determined by two initial conditions (ICs):

Apply IC #1: $u(x,0)=f(x)=\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi x}{L}\right)$

Multiply by mth eigenfunction and weighting function [$p(x)$ = 1 in this case] and integrate over interval of interest:

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx=\sum_{n=1}^{\infty}A_n\int_0^L\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$$

As we have already seen, this leads to

$$A_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Apply IC #2: $\left.\frac{\partial u}{\partial t}\right|_{t=0}=g(x)=\sum_{n=1}^{\infty}\left[-A_n\frac{n\pi v}{L}(0)+B_n\frac{n\pi v}{L}(1)\right]\sin\left(\frac{n\pi x}{L}\right)$

$$\text{or}\quad g(x)=\sum_{n=1}^{\infty}B_n\frac{n\pi v}{L}\sin\left(\frac{n\pi x}{L}\right)$$

Multiply by mth eigenfunction and weighting function [$p(x)$ = 1 again] and integrate over interval of interest to obtain

$$B_n=\frac{2}{L}\cdot\frac{L}{n\pi v}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx\quad\rightarrow\quad B_n=\frac{2}{n\pi v}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

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i. Full solution to the PDE is

$$u(x,t)=\sum_{n=1}^{\infty}\left[A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi vt}{L}\right)+B_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi vt}{L}\right)\right]$$

with $A_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$ and $B_n=\frac{2}{n\pi v}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$

j. As with heat equation, the BCs typically determine the spatial eigenfunctions and the ICs typically determine the coefficients (using the orthogonality of the inner products).

2. Interpretation of wave equation solution

   a. What can we learn from it?
   b. Vibration modes and resonances; harmonically related
   c. *Matlab* simulation
   d. Standing waves vs. traveling waves for $g(x) = 0$ (or $B_n = 0$) case. Use the identity

$$\sin(a)\cos(b)=\frac{1}{2}\sin(a+b)+\frac{1}{2}\sin(a-b)$$

to obtain

$$\begin{aligned}u(x,t)&=\sum_{n=1}^{\infty}\frac{A_n}{2}\left[\sin\left(\frac{n\pi x}{L}+\frac{n\pi vt}{L}\right)+\sin\left(\frac{n\pi x}{L}-\frac{n\pi vt}{L}\right)\right]\\&=\sum_{n=1}^{\infty}\frac{A_n}{2}\left\{\sin\left[\frac{n\pi}{L}(x+vt)\right]+\sin\left[\frac{n\pi}{L}(x-vt)\right]\right\}\end{aligned}$$