ENGR 695 Advanced Topics in Engineering Mathematics Fall 2023

Lecture Outline for Wednesday, Oct. 25

1. Continue with heat equation example:

$$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad \text{with} \quad u(0,t) = 0, \quad u(L,t) = 0, \quad \text{and} \quad u(x,0) = f(x)$$

   a. Review: Application of SOV method results in two linked ODEs:

$$X'' + \lambda X = 0 \quad \text{and} \quad T' + \lambda k T = 0$$

   b. Boundary conditions become

$$u(0,t) = X(0)T(t) = 0 \quad \rightarrow \quad X(0) = 0$$
$$u(L,t) = X(L)T(t) = 0 \quad \rightarrow \quad X(L) = 0$$

   c. The $X(x)$ problem is an S-L problem, and the $T(t)$ problem is an IVP with a first-order ODE:

$$X'' + \lambda X = 0 \quad \text{with} \quad X(0) = 0 \text{ and } X(L) = 0$$

$$T' + \lambda k T = 0$$

   d. Since the X problem has closed boundaries and is an S-L problem (homogeneous BCs), the most convenient solution form is

$$X(x) = c_1 \cos\left(\sqrt{\lambda}x\right) + c_2 \sin\left(\sqrt{\lambda}x\right)$$

   e. Apply BCs to obtain eigenfunctions and eigenvalues

$$X_n(x) = c_2 \sin\left(\frac{n\pi x}{L}\right) \quad \text{with} \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2$$

   f. Each eigensolution to the full problem (x and t domains) has the form:

$$u_n(x,t) = X(x)T(t) = A_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\lambda_n t}$$

   The unknown coefficient in the $T(t)$ solution has been subsumed into the composite constant A_n.

(*continued on next page*)

g. Each eigensolution u_n is a solution, but a linear combination of eigensolutions is also a solution. Full solution to the PDE for the general case:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) e^{-kn^2\pi^2 t/L^2}$$

How do we find the coefficients $\{A_n\}$? Next time!