ENGR 695 Advanced Topics in Engineering Mathematics Fall 2023

Lecture Outline for Monday, Dec. 4

1. Review/help session scheduling (final exam is 11:45 am – 2:45 pm Wednesday, Dec. 13):
   a. Review: Any time Monday, Dec. 11 or ending before 4:30 pm Tuesday, Dec. 12
   b. Tentative help session schedule:
      Monday, Dec. 11, 9:00–10:00 pm (Zoom)
      Tuesday, Dec. 12, 1:00–2:00 pm (BRKI 368) – depends on review session timing
      Wednesday, Dec. 13, 10:00–11:00 am (BRKI 368)

2. Review sheet for final exam posted soon. Exam covers material since Midterm Exam.

3. Correction to answer given for HW #8 Prob. 2 (Robin boundary condition in the Crank-Nicholson heat equation algorithm)

   a. Robin boundary condition:

$$\left.\frac{\partial u}{\partial x}\right|_{x=a} - h\,u(a,t) \approx \frac{u(a+\Delta x,\ j\Delta t) - u(a-\Delta x,\ j\Delta t)}{2\Delta x} - h\,u(a,\ j\Delta t) = -h\,u_m$$

   FD approximation simplifies to

$$u_{0,\,j} = u_{2,\,j} - 2\Delta x h\,u_{1,\,j} + 2\Delta x h\,u_m$$

   b. Regular update equation applied at location $i = 1$

$$u_{0,\,j+1} - \alpha u_{1,\,j+1} + u_{2,\,j+1} = -u_{0,\,j} + \beta u_{1,\,j} - u_{2,\,j}$$

   where

$$\alpha = 2\left(1 + \frac{\Delta x^2}{c\Delta t}\right) \qquad \text{and} \qquad \beta = 2\left(1 - \frac{\Delta x^2}{c\Delta t}\right)$$

   c. After substitution to eliminate $u_{0,j}$ and $u_{0,j+1}$:

$$\left(u_{2,\,j+1} - 2\Delta x h\,u_{1,\,j+1} + 2\Delta x h\,u_m\right) - \alpha u_{1,\,j+1} + u_{2,\,j+1} = -\left(u_{2,\,j} - 2\Delta x h\,u_{1,\,j} + 2\Delta x h\,u_m\right) + \beta u_{1,\,j} - u_{2,\,j}$$

$$\rightarrow \quad -\left(\alpha + 2\Delta x h\right)u_{1,\,j+1} + 2u_{2,\,j+1} = \left(\beta + 2\Delta x h\right)u_{1,\,j} - 2u_{2,\,j} - 4\Delta x h\,u_m,$$

not

$$-\left(\alpha + 2\Delta x h\right)u_{2,\,j+1} + 2u_{3,\,j+1} = \left(\beta + 2\Delta x h\right)u_{2,\,j} - 2u_{3,\,j} - 4\Delta x h\,u_m$$

(*continued on next page*)

d. The special update equation for the Robin BC at $x = a$ involves terms at $i = 1$ and $i = 2$, and the special update equation at $x = b$ involves terms at $i = N_x - 1$ and $i = N_x$. The total number of equations is therefore equal to N_x, which results in an $N_x \times N_x$ system of equations, compared to the $(N_x - 2) \times (N_x - 2)$ system for Dirichlet BCs. (The system for Neumann BCs is also $N_x \times N_x$ in size.)

e. Can express the system of equations for Robin BCs as

$$A\mathbf{u}_{j+1} = B\mathbf{u}_j + \mathbf{c} \quad \rightarrow \quad \mathbf{u}_{j+1} = A^{-1}B\mathbf{u}_j + A^{-1}\mathbf{c} \quad \rightarrow \quad \mathbf{u}_{j+1} = D\mathbf{u}_j + \mathbf{d},$$

where $D = A^{-1}B$, $\mathbf{d} = A^{-1}\mathbf{c}$,

$$A = \begin{bmatrix} -(\alpha+2\Delta xh) & 2 & 0 & 0 & \cdots & 0 \\ 1 & -\alpha & 1 & 0 & \cdots & 0 \\ 0 & 1 & -\alpha & 1 & & \vdots \\ \vdots & & & \ddots & & 0 \\ 0 & \cdots & 0 & 1 & -\alpha & 1 \\ 0 & \cdots & 0 & 0 & 2 & -(\alpha+2\Delta xh) \end{bmatrix}, \quad \mathbf{u}_{j+1} = \begin{bmatrix} u_{1,j+1} \\ u_{2,j+1} \\ u_{3,j+1} \\ \vdots \\ u_{N_x-1,j+1} \\ u_{N_x,j+1} \end{bmatrix}, \quad \mathbf{u}_j = \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ u_{3,j} \\ \vdots \\ u_{N_x-1,j} \\ u_{N_x,j} \end{bmatrix},$$

$$B = \begin{bmatrix} \beta+2\Delta xh & -2 & 0 & 0 & \cdots & 0 \\ -1 & \beta & -1 & 0 & \cdots & 0 \\ 0 & -1 & \beta & -1 & & \vdots \\ \vdots & & & \ddots & & 0 \\ 0 & \cdots & 0 & -1 & \beta & -1 \\ 0 & \cdots & 0 & 0 & -2 & \beta+2\Delta xh \end{bmatrix}, \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} -4\Delta xh\,u_m \\ 0 \\ 0 \\ \vdots \\ 0 \\ -4\Delta xh\,u_m \end{bmatrix}.$$

f. Not required in Prob.2, but for reference: Derivation of special update equation at $x = b$ for Robin BC (note sign changes; see p. 714 in Sec. 13.2 of Zill, 6th ed.):

$$\left.\frac{\partial u}{\partial x}\right|_{x=b} + hu(b,t) \approx \frac{u(b+\Delta x,\ j\Delta t) - u(b-\Delta x,\ j\Delta t)}{2\Delta x} + hu(b,\ j\Delta t) = hu_m$$

FD approximation simplifies to

$$\frac{u_{N_x+1,j} - u_{N_x-1,j}}{2\Delta x} + hu_{N_x,j} = hu_m \quad \rightarrow \quad u_{N_x+1,j} = u_{N_x-1,j} - 2\Delta xh\,u_{N_x,j} + 2\Delta xh\,u_m$$

At time $j + 1$:

$$u_{N_x+1,j+1} = u_{N_x-1,j+1} - 2\Delta xh\,u_{N_x,j+1} + 2\Delta xh\,u_m$$

(*continued on next page*)

Regular update equation applied at location $i = N_x$

$$u_{N_x-1,\,j+1} - \alpha u_{N_x,\,j+1} + u_{N_x+1,\,j+1} = -u_{N_x-1,\,j} + \beta u_{N_x,\,j} - u_{N_x+1,\,j}$$

Substitution to eliminate terms at $i = N_x + 1$:

$$u_{N_x-1,\,j+1} - \alpha u_{N_x,\,j+1} + \left(u_{N_x-1,\,j+1} - 2\Delta x h\, u_{N_x,\,j+1} + 2\Delta x h\, u_m\right) =$$
$$-u_{N_x-1,\,j} + \beta u_{N_x,\,j} - \left(u_{N_x-1,\,j} - 2\Delta x h\, u_{N_x,\,j} + 2\Delta x h\, u_m\right)$$

$$\rightarrow \quad 2u_{N_x-1,\,j+1} - \left(\alpha + 2\Delta x h\right) u_{N_x,\,j+1} = -2u_{N_x-1,\,j} + \left(\beta + 2\Delta x h\right) u_{N_x,\,j} - 4\Delta x h\, u_m$$