ENGR 695 Advanced Topics in Engineering Mathematics Fall 2023

Lecture Outline for Monday, Nov. 6

1. Wave equation (1-D) problems with open boundaries

$$v^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \qquad -\infty \le x \le \infty\,, \quad t \ge 0$$

   a. Special solution methods required
   b. One approach: See notes on D'Alembert's solution
   c. Solution has $x \pm vt$ in arguments of functions (traveling waves)
   d. Significance of parameter v

2. Wave equation in polar (cylindrical) coordinate system. Example: Vibrating circular membrane of radius c

$$a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) = \frac{\partial^2 u}{\partial t^2} \qquad \text{for} \quad 0 \le r \le c \quad \text{and} \quad t \ge 0$$

   a. ODEs after separation

$$r^2 R'' + rR' + \lambda r^2 R = 0 \quad \text{and} \quad T'' + \lambda a^2 T = 0$$

   b. Boundary conditions and initial conditions and their interpretation

$$u(c,t) = 0\,, \quad u(r,0) = f(r)\,, \quad \text{and} \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(r); \quad u \text{ is finite everywhere}$$

   c. Suggested solution forms for ODEs (with $\lambda_n = \alpha_n^2$)

$$R_n(r) = c_1 J_0(\alpha_n r) + c_2 Y_0(\alpha_n r) \quad \text{and} \quad T_n(t) = c_3 \cos(a\alpha_n t) + c_4 \sin(a\alpha_n t)$$

   d. Eigenvalues and eigensolutions (r_n = roots of J_0)

$$u_n(r,t) = \left[A_n \cos(a\alpha_n t) + B_n \sin(a\alpha_n t)\right] J_0(\alpha_n r) \quad \text{with} \quad \sqrt{\lambda_n} = \alpha_n = \frac{r_n}{c}$$

   e. In the case of a drum being struck by a stick or mallet, $f(r) = 0$ and $g(r)$ is a pulse centered at $r = 0$.
   f. Special additional condition: Solution must be finite within boundary (for $r \le c$)

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3. SOV solution and its interpretation

$$u(r,t)=\sum_{n=1}^{\infty}\left[A_n\cos(a\alpha_n t)+B_n\sin(a\alpha_n t)\right]J_0(\alpha_n r)$$

a. Inner product of Bessel functions; weighting function is $p(r) = r$

$$\left\langle J_0(\alpha_m r),J_0(\alpha_n r)\right\rangle=\int_0^c r\,J_0(\alpha_m r)J_0(\alpha_n r)\,dr$$

b. Apply ICs to find expressions for $\{A_n\}$ and $\{B_n\}$ coefficients

$$u(r,0)=f(r)=\sum_{n=1}^{\infty}\left[A_n(1)+B_n(0)\right]J_0(\alpha_n r)=\sum_{n=1}^{\infty}$$

$$\rightarrow\quad\int_0^c f(r)J_0(\alpha_m r)\,r\,dr=\sum_{n=1}^{\infty}A_n\int_o^c J_0(\alpha_n r)J_0(\alpha_m r)\,r\,dr\quad\rightarrow\quad A_n=\frac{\left\langle f(r),J_0(\alpha_n r)\right\rangle}{\left\|J_0(\alpha_n r)\right\|^2}$$

$$\frac{\partial u}{\partial t}=\sum_{n=1}^{\infty}\left[-A_n a\alpha_n\sin(a\alpha_n t)+B_n a\alpha_n\cos(a\alpha_n t)\right]J_0(\alpha_n r)$$

$$\rightarrow\quad B_n=\frac{\left\langle f(r),J_0(\alpha_n r)\right\rangle}{a\alpha_n\left\|J_0(\alpha_n r)\right\|^2}$$

Note that $\left\|J_0(\alpha_n r)\right\|^2=\dfrac{c^2}{2}J_1^2(\alpha_n c)$

c. What does the result mean? How are the eigenvalues used and interpreted?
d. *Matlab* simulation
e. Are there standing waves? How do they compare to vibrating string case?