ENGR 695 Advanced Topics in Engineering Mathematics Fall 2023

Lecture Outline for Friday, Oct. 6

1. Fourier equation as an eigenvalue problem; three possible solution forms (for closed boundaries)

$$y'' + \lambda y = 0$$

   a. If $\lambda < 0$: $y(x) = c_1 \cosh\left(\sqrt{-\lambda}x\right) + c_2 \sinh\left(\sqrt{-\lambda}x\right)$
   b. If $\lambda = 0$: $y(x) = c_1 + c_2 x$
   c. If $\lambda > 0$: $y(x) = c_1 \cos\left(\sqrt{\lambda}x\right) + c_2 \sin\left(\sqrt{\lambda}x\right)$

2. Other ODE classes with variable coefficients (2^{nd} order only)

   a. Parametric Bessel equation of order ν (parameter is α)

$$x^2 y'' + xy' + \left(\alpha^2 x^2 - \nu^2\right) y = 0$$

   b. Modified parametric Bessel equation of order ν

$$x^2 y'' + xy' - \left(\alpha^2 x^2 + \nu^2\right) y = 0$$

   c. Legendre equation of order n

$$\left(1 - x^2\right) y'' - 2xy' + n(n+1) y = 0$$

   d. Airy equation

$$y'' \pm a^2 xy = 0$$

   e. Chebyshev equation

$$\left(1 - x^2\right) y'' - xy' + a^2 y = 0$$

(*continued on next page*)

3. Solutions to DEs with variable coefficients – the general approach:

    a. Power series solution

$$y(x)=\sum_{n=0}^{\infty}c_n\left(x-x_o\right)^n,$$

    but most of the ones we will see are expanded about $x_o = 0$:

$$y(x)=\sum_{n=0}^{\infty}c_n x^n$$

    b. Ordinary points: Values of x at which variable coefficients are analytic
    c. Singular points: Values of x that are not ordinary points
    d. Singular points can regular or irregular
    e. Example: $\ln(x)$ is not analytic for $x \le 0$, so the ordinary points are $x > 0$
    f. Power series solutions valid only over intervals for which they converge; they are often not valid over all of $-\infty < x < \infty$

4. Example: Parametric Bessel equation of order ν

$$x^2y''+xy'+\left(\alpha^2x^2-\nu^2\right)y=0$$

    a. The "parameter" is α
    b. Series solution (just one of the two possible solutions) is the *Bessel function of the first kind*,

$$J_\nu\left(\alpha x\right)=\sum_{n=0}^{\infty}\frac{\left(-1\right)^n}{n!\Gamma\left(1+\nu+n\right)}\left(\frac{\alpha x}{2}\right)^{2n+\nu},$$

    where $\Gamma(1+\nu)$ is the gamma function (more info in Appendix II of Zill, 6th ed.)
    c. Second series solution, which is LI from J_ν if $\nu \neq$ integer:

$$J_{-\nu}\left(\alpha x\right)=\sum_{n=0}^{\infty}\frac{\left(-1\right)^n}{n!\Gamma\left(1-\nu+n\right)}\left(\frac{\alpha x}{2}\right)^{2n-\nu}$$

    d. Many practical applications with $\nu =$ integer. A more general second solution is the *Bessel function of the second kind*,

$$Y_\nu\left(\alpha x\right)=\frac{\cos\left(\nu x\right)J_\nu\left(\alpha x\right)-J_{-\nu}\left(\alpha x\right)}{\sin\left(\nu x\right)}.$$

    Seems to be indeterminate for certain values of x when $\nu =$ integer (get 0/0). However, it can be shown that Y_ν exists and is LI from J_ν even if ν is an integer.

e. Thus, the general solution of the parametric Bessel equation can be written

$$y(x) = c_1 J_\nu(\alpha x) + c_2 Y_\nu(\alpha x)$$

f. The J_ν and Y_ν functions are well known and tabulated in old references (see web link: NIST Digital Library of Mathematical Functions), but most modern math software packages have Bessel functions built in (e.g., in *Matlab*, they are `besselj` and `bessely`).

g. See “Meet the Bessels” by Prof. Maneval for info on Bessel functions of zero order.