1. (a) | Step | \( x \) | approximate y-value | \( \Delta y \) = slope \cdot \Delta x |
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>((1+0.1)(1) = .1)</td>
</tr>
<tr>
<td>1</td>
<td>0.1</td>
<td>1 + .1 = 1.1</td>
<td>((1.1+0.1(1.1))(1) = .121)</td>
</tr>
<tr>
<td>2</td>
<td>0.2</td>
<td>1.1 + .121 = 1.221</td>
<td>((1.221 + 0.2(1.221))(1) = .14652)</td>
</tr>
<tr>
<td>3</td>
<td>0.3</td>
<td>1.221 + .14652 (\approx 1.368)</td>
<td>((1.368 + 0.3(1.368))(1) \approx .178)</td>
</tr>
<tr>
<td>4</td>
<td>0.4</td>
<td>1.368 + .178 = 1.546</td>
<td>((1.546 + 0.4(1.546))(1) \approx .216)</td>
</tr>
<tr>
<td>5</td>
<td>0.5</td>
<td>1.546 + .216 = 1.762</td>
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\( y(0.5) \approx 1.762 \)

(b) The slope field shows that the solution curves are concave up in the first quadrant, so a soln. curve through (0,1) will be concave up to the right of the y-axis. Thus the tangent lines will lie beneath the soln., and our estimate is an underestimate.

(c) We have

\[
\frac{dy}{dx} = y + xy = y(1+x) \\
\int \frac{dy}{y} = \int (1+x) \, dx \quad (y \neq 0) \\
\ln |y| = x + \frac{1}{2} x^2 + C \\
|y| = e^{x + \frac{1}{2} x^2 + C} = e^C e^{x + \frac{1}{2} x^2}
\]

We check that \( y = 0 \) is also a solution, so \( y = Ae^{x + \frac{1}{2} x^2} \), A any constant, is the general solution. Plugging in our initial condition gives

\[
1 = Ae^{0} \Rightarrow A = 1,
\]

so the exact solution is \( y = e^{x + \frac{1}{2} x^2} \). We have \( y(0.5) = e^{0.5 + 0.25(0.5)^2} \approx 1.868 \). Thus the error in using our approx. in (a) is \(1.868 - 1.762 = .106\).

2. (a) We know the Taylor series for \( e^t \) about \( t = 0 \) is \( \sum_{n=0}^{\infty} \frac{t^n}{n!} \), so the Taylor series for \( te^t \) about \( t = 0 \) is

\[
t \cdot \sum_{n=0}^{\infty} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{t^{n+1}}{n!}.
\]

The general term is \( \frac{t^{n+1}}{n!} \).

(b) We have

\[
\int_0^x te^t \, dt = \left[ te^t - \frac{t^2}{2} + \frac{t^3}{3!} + \frac{t^4}{4!} + \ldots \right]_0^x = \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!(n+2)}
\]
(c) We note that the sum \( \frac{1}{2} + \frac{1}{3} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \ldots \) is precisely the Taylor series we found in (b) with \( x = 1 \). So

\[
\text{sum} = \int_0^1 \te^t \, dt = \left[ \te^t - \te^t \right]_0^1 = e - e - (0 - 1) = 1.
\]

(3a) Let \( f(x) = \sin x \). Then

\[
\begin{align*}
f(-\frac{\pi}{3}) &= \sin(-\frac{\pi}{3}) = -\frac{\sqrt{3}}{2} \\
f'(-\frac{\pi}{3}) &= \cos(-\frac{\pi}{3}) = \frac{1}{2} \\
f''(-\frac{\pi}{3}) &= -\sin(-\frac{\pi}{3}) = \frac{\sqrt{3}}{2} \\
f'''(-\frac{\pi}{3}) &= -\cos(-\frac{\pi}{3}) = -\frac{1}{2}
\end{align*}
\]

and the Taylor polynomial of degree 3 approx. \( \sin x \) for \( x \) near \( a = -\frac{\pi}{3} \) is

\[
P_3(x) = -\frac{\sqrt{3}}{2} + \frac{1}{2} (x + \frac{\pi}{3}) + \frac{\sqrt{3}}{2(2!)} (x + \frac{\pi}{3})^2 - \frac{1}{2(3!)} (x + \frac{\pi}{3})^3.
\]

(b) We have \( f^{(4)}(x) = \sin x \), and this fcn. is maximized on \([-\frac{\pi}{2}, -\frac{\pi}{3}]\) at \( x = -\frac{\pi}{2} \), where it has value \( \sin(-\frac{\pi}{2}) = -1 \). Thus \( M = 1 \). The Lagrange error bound is

\[
\text{Error} \leq \frac{1}{4!} \left| -\frac{\pi}{2} + \frac{\pi}{3} \right|^4 = \frac{1}{4!} \cdot \left( \frac{\pi}{3} \right)^4.
\]

4. (a) Suppose that a straight line is a soln. to the diff. eqn. \( \frac{dy}{dx} = x - \frac{1}{2} y \).

Since the general eqn. for a straight line is \( y = ax + b \), we can plug the appropriate pieces into our diff. eqn. and solve for \( a \) and \( b \):

\[
\begin{align*}
y &= ax + b \Rightarrow \frac{dy}{dx} &= a \\
\frac{dy}{dx} &= x - \frac{1}{2} y &\Rightarrow a = x - \frac{1}{2} (ax + b) &\Rightarrow a = x(1-\frac{a}{2}) - \frac{b}{2} \\
1 - \frac{a}{2} &= 0 &\text{and} &\quad a = -\frac{b}{2} \\
a &= 2 &\Rightarrow b &= -4
\end{align*}
\]

So if \( y = ax + b \) is to be a soln. to our diff. eqn., it must be the line \( y = 2x - 4 \).

(b) (c) many possible soln. curves; one is sketched on slope field