Building polygons from spectral data

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Joint work with Victor Guillemin and Rosa Sena-Dias
Outline

1. Motivation
2. The Problem
3. Some Solutions
4. Implications
Vibrating Drumheads

An example of the Laplacian on a domain $D$ in the Euclidean plane.

Vibration frequencies $\leftrightarrow$ Eigenvalues of $\Delta$ on $D$

How much geometry is encoded in the spectrum?
Abreu: Can one hear the shape of a Delzant polytope?
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A convex polytope $P$ in $\mathbb{R}^n$ is *Delzant* if

- it is *simple*, i.e., there are $n$ facets meeting at each vertex;
- it is *rational*, i.e., for every facet of $P$, a primitive outward normal can be chosen in $\mathbb{Z}^n$;
- it is *smooth*, i.e., for every vertex of $P$, the outward normals corresponding to the facets meeting at that vertex form a basis for $\mathbb{Z}^n$. 
Examples and Non-examples

(0,1) (0,0) (1,0)

(0,1) (0,0) (2,0)

(0,0)
So What?

Symplectic geometers care about Delzant polytopes because...
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- $M^{2n}$ is toric manifold, i.e., symplectic manifold with “compatible” $\mathbb{T}^n$-action
- $g$, toric Kähler metric on $M$
- Delzant/moment polytope associated to $M$
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Abreu’s question: Let $M$ be a toric manifold equipped with a toric Kähler metric $g$. Does the spectrum of the Laplacian $\Delta_g$ determine the moment polytope of $M$?
Modifying Abreu’s Question: Step 1

A convex polytope $P$ in $\mathbb{R}^n$ is \textit{rational simple} if it is simple, it is rational, and for every vertex of $P$, the outward normals corresponding to the facets meeting at that vertex form a basis for $\mathbb{Q}^n$. 
Modifying Abreu’s Question: Step 1

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![Diagram of a triangle with vertices at (0,0), (2,0), and (0,1)]
Modifying Abreu’s Question: Step 2

Equivariant spectrum = Laplace spectrum + weights for each eigenvalue
Motivation
The Problem
Some Solutions
Implications

Modifying Abreu’s Question: Step 2

Equivariant spectrum $=\text{ Laplace spectrum} + \text{ weights for each eigenvalue}$

Question

Let $M$ be a toric orbifold equipped with a toric Kähler metric $g$. Does the equivariant spectrum of the Laplacian $\Delta_g$ determine the labeled moment polytope of $M$?
The equivariant spectrum associated to a toric orbifold $M$ whose moment polytope has no parallel facets determines:

1. the (unsigned) normal directions to the facets;
2. the volumes of the corresponding facets;
3. the labels of the facets.
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1. the (unsigned) normal directions to the facets;
2. the volumes of the corresponding facets;
3. the labels of the facets.

Given this data, how many labeled moment polytopes can you build?
Building Polygons
Minkowski’s Theorem

(Minkowski; Klain) Given a list
\{(u_i, \nu_i), u_i \in \mathbb{R}^n, \nu_i \in \mathbb{R}^+, \ i = 1, \ldots, d\} where the u_i are unit vectors that span \mathbb{R}^n, there exists a convex polytope P with facet normals u_1, \ldots, u_d and corresponding facet volumes \nu_1, \ldots, \nu_d if and only if

\[ \sum_{i=1}^{d} \nu_i u_i = 0. \]

Moreover, this polytope is unique up to translation.
Troublemaker 1: subpolytopes

Lemma
Let $P$ be a convex polytope in $\mathbb{R}^n$ with no subpolytopes and facet volumes $\nu_1, \ldots, \nu_d$. Assume that the facet normals to $P$ are $u_1, \ldots, u_d$ up to sign. Then, up to translation, there are only $2$ choices for the set of signed normals.
Lemma

Let $P$ be a convex polytope in $\mathbb{R}^n$ with no subpolytopes and facet volumes $\nu_1, \ldots, \nu_d$. Assume that the facet normals to $P$ are $u_1, \ldots, u_d$ up to sign. Then, up to translation, there are only 2 choices for the set of signed normals.
Troublemaker 2: parallel facets

Parallel facets introduce indeterminants:

- know *sum* of volumes of facets in parallel pair
- do not know which normal directions in list are repeated
Motivation

The Problem

Some Solutions

Implications

Bye-bye, troublemaking polytopes

Lemma

Close to any rational simple polytope in $\mathbb{R}^n$, there is a rational simple polytope that has no parallel facets and has no subpolytopes.
Bye-bye, troublemaking polytopes

Lemma

*Close to any rational simple polytope in $\mathbb{R}^n$, there is a rational simple polytope that has no parallel facets and has no subpolytopes.*

Orbifolds are better than manifolds!
Question

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Theorem

(D–V. Guillemin–R. Sena-Dias) Let $M$ be a generic toric orbifold with a fixed torus action and a toric Kähler metric. Then the equivariant spectrum of $M$ determines the labeled moment polytope $P$ of $M$, up to two choices and up to translation.
Abreu’s original question

Question

*Can one hear the shape of a Delzant polytope?*
Abreu’s original question

**Question**

*Can one hear the shape of a Delzant polytope?*

**Theorem (D–V. Guillemin–R. Sena-Dias)**

*Let $M^4$ be a toric symplectic manifold with a fixed torus action and a toric metric. Given the equivariant spectrum of $M$ and the spectrum of the associated real manifold, we can reconstruct the moment polygon $P$ of $M$ up to two choices and up to translation for generic polygons with no more than 2 pairs of parallel sides.*
Open questions and future directions

- Can the two possibilities be distinguished using spectral data?
- Are the genericity assumptions necessary?
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- Are the genericity assumptions necessary?
- What can we say about the metric?
- Can one hear the shape of a Delzant polytope?!