Eigenvalue (mis)behavior on manifolds

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Outline

1. Isoperimetric inequalities
2. Upper bounds on eigenvalues for manifolds
3. Metrics invariant under a group action
4. Submanifolds
The Original Isoperimetric Inequality

The Problem of Queen Dido: maximize the size of Carthage
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- What about \textit{closed} curves?
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  - planar
  - simple
  - fix length $L$, maximize area $A$
The Original Isoperimetric Inequality

- The Problem of Queen Dido: maximize the size of Carthage
- What about *closed* curves?
  - planar
  - simple
  - fix length $L$, maximize area $A$
  - “The” isoperimetric inequality:

\[ L^2 \geq 4\pi A \]
Generalizations

- $\mathbb{R}^n$: minimize surface area among domains with fixed volume
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- $\mathbb{R}^n$: minimize surface area among domains with fixed volume
- Mathematical physics: a physical quantity is extremal for a circular or spherical domain
An example

Setup:

- domain $D \subset \mathbb{R}^2$
- $f : D \rightarrow \mathbb{R}$, a smooth function which equals zero on the boundary of $D$
- $\Delta f := \frac{\partial^2 f}{\partial^2 x} + \frac{\partial^2 f}{\partial^2 y}$
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Seek solutions to $\Delta f = \lambda f$

Especially interested in $\lambda_1$
The Rayleigh quotient for domains

**Theorem**

Let $D$ be a domain with $\Delta$ acting on piecewise smooth, nonzero functions $f$ which are zero on the boundary of $D$, and with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots$. For any such $f$,

$$\lambda_1 \leq \frac{\int_D |\nabla f|^2}{\int_D f^2},$$

with equality if and only if $f$ is an eigenfunction of $\lambda_1$. 
Minima of the Rayleigh quotient

Theorem (Rayleigh, Faber-Krahn)

Among all domains $D \subset \mathbb{R}^2$ with fixed area, the infimum of the Rayleigh quotient attains a minimum if and only if $D$ is a circular disk.
Minima of the Rayleigh quotient

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Higher-dimensional analog: Rayleigh quotient attains minimum iff $D \subset \mathbb{R}^n$ is sphere.
The Rayleigh quotient for manifolds

Setup:

- \((M, g)\), compact Riemannian manifold
- \(\Delta\), Laplace operator on \((M, g)\)
- Eigenvalues of \(\Delta\) are

\[0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots\]
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- Rayleigh quotient:

\[
\lambda_1(M) = \inf_{f \in \mathcal{F}_1} \frac{\int_M |\nabla f|^2}{\int_M f^2},
\]

where \(\mathcal{F}_1\) is set of smooth nonzero functions on \(M\) orthogonal to the constant functions
Hersch’s Theorem

Theorem (Hersch)

Consider the sphere $S^2$ equipped with any Riemannian metric $g$. We have

$$\lambda_1 \text{Vol}(g) \leq 8\pi,$$

with equality only in the case of the constant curvature metric.

Idea of proof: Move $S^2$ to its center of mass, and use coordinate functions as test functions in the Rayleigh quotient.
Compact orientable surfaces

Theorem (Yang-Yau)

Let \((M, g)\) be a compact orientable surface of genus \(\gamma\). Then

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\lambda_1(g) \text{Vol}(g) \leq 8\pi \left\lfloor \frac{\gamma + 3}{2} \right\rfloor.
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Generalized to nonorientable surfaces by Li-Yau
Theorem (Korevaar)

Let \((M, g)\) be a compact orientable surface of genus \(\gamma\), and let \(C > 0\) be a universal constant. For every integer \(k \geq 1\),

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\lambda_k(g) \Vol(g) \leq C(\gamma + 1)k.
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Open questions abound, e.g., optimal bound for \(\lambda_2\) on Klein bottle or surface of genus 2.
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**Theorem (Colbois-Dodziuk)**

Let $(M^n, g)$ be a compact, closed, connected manifold of dimension at least three. Then

$$\sup \lambda_1(g) \frac{\text{Vol}(g)^{2/n}}{= \infty},$$

where the supremum is taken over all Riemannian metrics $g$ on $M$. 
Idea of proof

- Use Bleecker’s result: take \((S^n, g_0)\) such that \(\text{Vol}(S^n, g_0) = 1\) and \(\lambda_1(g_0) \geq k + 1\), where \(k\) is a large constant.
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- Connected sum is diffeomorphic to \(M\), contains submanifold \(\Omega\) naturally identified with \(S^n \setminus B_\rho\).
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- Take arbitrary metric \(g_1\) on \(M\) whose restriction to \(\Omega\) equals \(g_0\) restricted to \(\Omega\), make \(g_1\) really small on most of \(M \setminus \Omega\) without changing it on \(\Omega\).
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- \(M\) “looks like” \((S^n, g_0)\), and \(\lambda_1\) for modified \(g_1\) is like \(\lambda_1(g_0)\).
Where do we go from here?

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- **intrinsic constraints**: restrict to conformal class of metrics, to projective Kähler metrics, to metrics which preserve the symplectic or Kähler structure, etc.

- **extrinsic constraints**: mean curvature (Reilly’s inequality)
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- consider the subset of $S^1$-invariant metrics
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Back to the 2-sphere

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Bound is attained by the union of two disks of equal area
What happens for higher-dimensional spheres?

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**Theorem (Colbois-D-El Soufi)**

Let $(S^n, g)$ be as above, with $\text{Vol}(g) = 1$. Then, for all $k \in \mathbb{Z}$,

$$\lambda_k^{O(n)}(g) < \lambda_k^{O(n)}(D^n) \text{Vol}(D^n)^{2/n},$$

where $D^n$ is the Euclidean $n$-ball of volume $1/2$. 

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What about *any* manifold, not just spheres?

- replace $S^n$ by ccc manifold $M$ of dimension $n \geq 3$
- replace $O(n)$ by finite subgroup $G$ of group of diffeomorphisms acting on $M$
- let $\Delta$ act on $G$-invariant functions
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Then $\lambda_1^G(g)\text{Vol}(g)^{2/n}$ is unbounded!

Proof: apply Colbois-Dodziuk “equivariantly”
Dropping one hypothesis

- ccc manifold $M$ of dimension $n \geq 3$
- discrete group $G$ acting on $M$
- consider $G$-invariant metrics on $M$
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Question: Does $\lambda_1(g)\text{Vol}(g)^{2/n}$ become arbitrarily large?
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Question: Does $\lambda_1(g)\text{Vol}(g)^{2/n}$ become arbitrarily large?

(Partial) Answer: Work of Paul Cernea
An extrinsic constraint

Hypersurfaces: curve in plane, two-dimensional surface in $\mathbb{R}^3$

Submanifolds: equator in $S^2$, manifold in $\mathbb{R}^k$ for $k$ sufficiently large
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Why *extrinsic*?
Theorem (Colbois-D-El Soufi)

Let $M$ be a compact convex hypersurface in $\mathbb{R}^{n+1}$. Then

$$\lambda_1(M) \frac{\text{Vol}(M)^{2/n}}{n} \leq A(n) \lambda_1(S^n) \frac{\text{Vol}(S^n)^{2/n}}{n},$$

where $\lambda_1(S^n) = n$ and $A(n) = \frac{(n+2) \text{Vol}(S^n)}{2 \text{Vol}(S^{n-1})}$.

Why is there no mention of a metric?
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Proof uses barycentric methods and projection
Replacing “convex”

Hypersurface $M$: intersection index is maximum number of collinear points in $M$
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Submanifold $M^n$ in $\mathbb{R}^{n+p}$: intersection index of $M$ is

$$i(M) = \sup_{\Pi} \# M \cap \Pi,$$

where $\Pi$ runs over set of $p$-planes transverse to $M$ in $\mathbb{R}^{n+p}$
Theorem (Colbois-D-El Soufi)

Let $M^n$ be a compact immersed submanifold of a Euclidean space $\mathbb{R}^{n+p}$. Then

$$\lambda_1(M) \Vol(M)^{2/n} \leq A(n) \left( \frac{i(M)}{2} \right)^{1+\frac{2}{n}} \lambda_1(S^n) \Vol(S^n)^{2/n}.$$
Theorem (Colbois-D-El Soufi)

For every compact $n$-dimensional immersed submanifold $M$ of $\mathbb{R}^{n+p}$ and for every integer $k$,

$$\lambda_k(M) \text{Vol}(M)^{2/n} \leq c(n)i(M)^{2/n}k^{2/n},$$

where $c(n)$ is an explicit constant depending only on the dimension $n$. 
What does it all mean?

Combining Colbois-Dodziuk with our results in the extrinsic context says...
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Given a smooth manifold $\tilde{M}$ of dimension $n \geq 3$, there exist Riemannian metrics $g$ of volume 1 on $\tilde{M}$ such that any immersion of $\tilde{M}$ into a Euclidean space $\mathbb{R}^{n+p}$ which preserves $g$ must have a very large intersection index and volume which concentrates into a small Euclidean ball.
Summary

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- For manifolds of dimension at least three, getting bounds on $\lambda_1$ requires adding more constraints, either intrinsic (like invariance of the metric and eigenfunctions under a group action) or extrinsic (like immersed submanifolds).
Summary

- One physical isoperimetric problem is to extremize $\lambda_1$ subject to certain constraints, the most basic of which is the volume of the manifold.
- The Rayleigh quotient and spheres often play key roles in the solutions to this isoperimetric problem.
- For manifolds of dimension at least three, getting bounds on $\lambda_1$ requires adding more constraints, either intrinsic (like invariance of the metric and eigenfunctions under a group action) or extrinsic (like immersed submanifolds).

Outlook
- Are there other natural constraints, either of an intrinsic or extrinsic nature, that give interesting results?
- When upper bounds exist, can we show that they are optimal?
References

