Extremal invariant eigenvalues of the Laplacian of invariant metrics

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8 January 2008

The Plan

1. Historical motivation
2. S^1-invariant eigenvalues on S^2
3. Lie groups acting on manifolds
4. Future directions

Setup

M: compact connected Riemannian manifold of dimension $n \geq 2$

g: Riemannian metric on M, with associated Laplacian Δ_g and its spectrum

$Spec(g) = \{0 = \lambda_0(g) < \lambda_1(g) \leq \ldots \leq \lambda_k(g) \leq \ldots\}$

Question Consider the kth eigenvalue, normalised as a functional

$$g \to \lambda_k(g)\mathrm{Vol}(g)^{2/n}$$

on the space of Riemannian metrics on M. Are there extremal metrics for this functional?

Sampling of Answers for Surfaces

S^2 [Hersch] :

$$\lambda_1(g)\text{Vol}(g) \leq 8\pi = \lambda_1(g_{\text{can}})\text{Vol}(g_{\text{can}})$$

Orientable surface of genus γ [Yang-Yau]:

$$\lambda_1(g)\text{Vol}(g) \leq 8\pi \left[\frac{\gamma+3}{2}\right]$$

Surface of genus γ [Korevaar]: There exists a universal constant $C > 0$ such that for all $k > 0$,

$$\lambda_k(g) \leq Ck(\gamma+1).$$

Higher Dimensions

THEOREM *[Colbois-Dodziuk]: $dim(M) \geq 3$*

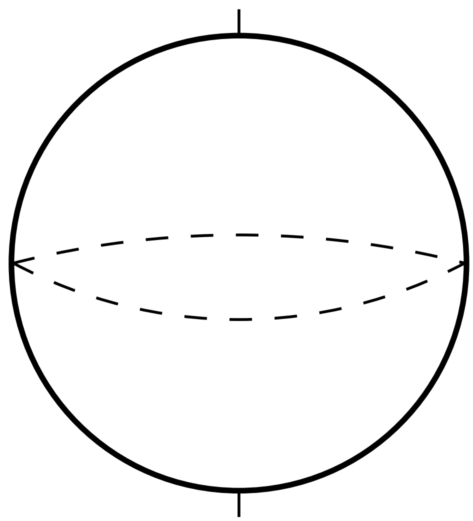
$$\sup_g \{\lambda_1(g) Vol(g)^{2/n}\} = \infty$$

To study extremal properties of spectrum, we need to add some restrictions...

- conformal class (Korevaar, El Soufi-Ilias)
- projective Kähler metrics (Bourguignon-Li-Yau)
- symplectic or Kähler metrics (L. Polterovich)
- invariance under isometries (Abreu-Freitas)

S^1 acts on S^2

Consider S^2 with metrics which are smooth, have total area 4π, and are S^1-invariant.



Denote the invariant eigenvalues by $\lambda_k^{\text{inv}}(g)$.

THEOREM *[Abreu-Freitas]: In this setting,* $\lambda_1^{\text{inv}}(g)$ *can be any number strictly between* 0 *and* ∞.

Do more restrictions help get bounds?

Fixed Gauss curvature at poles: still have $0 < \lambda_1^{\mathrm{inv}}(g) < \infty$

Metrics embedded in $\mathbb{R}^3$:

$$\lambda_k^{\mathrm{inv}}(g) < \frac{1}{2}\xi_k^2$$

and in particular

$$\lambda_1^{\mathrm{inv}}(g) < \frac{1}{2}\xi_1^2 \approx 2.89$$

Can characterize the supremum geometrically

Questions, Questions, Questions

- What can we say about the functional $\lambda_k^{inv}(g)\mathrm{Vol}(g)^{2/n}$ for a general compact differentiable G-manifold?
- Does being embedded guarantee a bound on $\lambda_k^{\mathrm{inv}}(g)\mathrm{Vol}(g)^{2/n}$?
- If we find critical metrics, to what do they correspond geometrically?

Restrictions: Invariance and Conformal Class

Assumptions

- *dim*$(M) \geq 3$
- *G: Lie group of dimension ≥ 1 acting effectively on M by isometries*
- *dim*$(M/G) \geq 1$

Theorem *[Colbois-D-El Soufi]: Let (M, g_0) and G be as above. Then*

$$\sup_g \{\lambda_1^{inv}(g)\, Vol(g)^{2/n}\} = \infty,$$

where the metrics g are G-invariant and conformal to g_0.

Korevaar Revisited

THEOREM *[Korevaar]:*

$$\lambda_k(g)\,Vol(g)^{2/n} \;\leq C_n([g_0])k^{2/n},$$

for any g *conformal to* g_0*;* C *depends only on* n *and the conformal class* $[g_0]$ *of* g_0*.*

COROLLARY *Let* (M, g_0) *and* G *be as on preceding slide. For any positive integer* N*, there exists a* G*-invariant metric* g_N *conformal to* g_0 *such that none of the first* N *eigenfunctions of* g_N *is* G*-invariant.*

Remark The assumption on the dimension of G is necessary in theorem on preceding slide. We can remove the conformal requirement and recover the same result for G a discrete group.

Restrictions: Invariance and Embedded

Let g be an $O(n)$-invariant metric of volume 1 on S^n embedded as a hypersurface of revolution in $\mathbb{R}^{n+1}$.

THEOREM *[Colbois-D-El Soufi]: For all k,*

$$\lambda_k^{inv}(g) < \lambda_k^{inv}(D^n)\,Vol(D^n)^{2/n}.$$

Furthermore, there exists a sequence g_i of $O(n)$-invariant metrics on S^n in $\mathbb{R}^{n+1}$ with

$$\lambda_k^{inv}(g_i)\,Vol(g_i)^{2/n} \to \lambda_k^{inv}(D^n)\,Vol(D^n)^{2/n},$$

but $\lambda_k^{inv}(D^n)\,Vol(D^n)^{2/n}$ is not attained by any smooth metric on S^n.

Embedding is not enough!

PROPOSITION *[Colbois-D-El Soufi]: Within the class of smooth S^1-invariant metrics g on T^2 which correspond to an embedding of T^2 in $\mathbb{R}^3$,*

$$\sup_g \{\lambda_1^{inv}(g) Vol(g)\} = \infty.$$

Remark The argument also works for a general torus $T^{n+1} = S^1 \times S^n$.

Future Directions

- Can we construct G-invariant metrics, G discrete, such that $\lambda_1(g)\mathrm{Vol}(g)^{2/n}$ gets arbitrarily large?
- What happens if we look at invariant p-forms, $p > 0$?
- For every $k \in \mathbb{N}$, there exists an integer $m(k, g) \geq k$ such that

  $$\lambda_k^{\mathrm{inv}}(g) = \lambda_{m(k,g)}(g).$$

  What is the behavior of $m(k, g)$?