Extremal invariant eigenvalues of the Laplacian of invariant metrics

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The Plan

1. Historical motivation
2. $S^1$-invariant eigenvalues on $S^2$
3. Lie groups acting on manifolds
4. Future directions
Setup

\(M\): compact connected Riemannian manifold of dimension \(n \geq 2\)

\(g\): Riemannian metric on \(M\), with associated Laplacian \(\Delta_g\) and its spectrum

\(Spec(g) = \{0 = \lambda_0(g) < \lambda_1(g) \leq \ldots \leq \lambda_k(g) \leq \ldots\}\)

**Question**  Consider the \(k\)th eigenvalue, normalised as a functional

\[g \rightarrow \lambda_k(g)\text{Vol}(g)^{2/n}\]

on the space of Riemannian metrics on \(M\). Are there extremal metrics for this functional?
Sampling of Answers for Surfaces

\( S^2 \) [Hersch]:
\[
\lambda_1(g) \text{Vol}(g) \leq 8\pi = \lambda_1(g_{\text{can}}) \text{Vol}(g_{\text{can}})
\]

**Orientable surface of genus** \( \gamma \) [Yang-Yau]:
\[
\lambda_1(g) \text{Vol}(g) \leq 8\pi \left[ \frac{\gamma + 3}{2} \right]
\]

**Surface of genus** \( \gamma \) [Korevaar]: There exists a universal constant \( C > 0 \) such that for all \( k > 0 \),
\[
\lambda_k(g) \leq Ck(\gamma + 1).
\]
Higher Dimensions

**Theorem** [Colbois-Dodziuk]: \( \dim(M) \geq 3 \)

\[
\sup_g \{ \lambda_1(g) \text{Vol}(g)^{2/n} \} = \infty
\]

To study extremal properties of spectrum, we need to add some restrictions...

- conformal class (Korevaar, El Soufi-Ilias)
- projective Kähler metrics (Bourguignon-Li-Yau)
- symplectic or Kähler metrics (L. Polterovich)
- invariance under isometries (Abreu-Freitas)
$S^1$ acts on $S^2$

Consider $S^2$ with metrics which are smooth, have total area $4\pi$, and are $S^1$-invariant.

Denote the invariant eigenvalues by $\lambda_{k}^{\text{inv}}(g)$.

**Theorem [Abreu-Freitas]:** In this setting, $\lambda_1^{\text{inv}}(g)$ can be any number strictly between 0 and $\infty$. 
Do more restrictions help get bounds?

Fixed Gauss curvature at poles: still have
0 < \( \lambda_1^{\text{inv}}(g) \) < \( \infty \)

Metrics embedded in \( \mathbb{R}^3 \):
\[
\lambda_k^{\text{inv}}(g) < \frac{1}{2} \xi_k^2
\]
and in particular
\[
\lambda_1^{\text{inv}}(g) < \frac{1}{2} \xi_1^2 \approx 2.89
\]

Can characterize the supremum geometrically
Questions, Questions, Questions

- What can we say about the functional $\lambda_k^{\text{inv}} (g) \text{Vol}(g)^{2/n}$ for a general compact differentiable $G$-manifold?
- Does being embedded guarantee a bound on $\lambda_k^{\text{inv}} (g) \text{Vol}(g)^{2/n}$?
- If we find critical metrics, to what do they correspond geometrically?
Restrictions: Invariance and Conformal Class

ASSUMPTIONS

• $\dim(M) \geq 3$

• $G$: Lie group of dimension $\geq 1$ acting effectively on $M$ by isometries

• $\dim(M/G) \geq 1$

THEOREM [Colbois-D-El Soufi]: Let $(M, g_0)$ and $G$ be as above. Then

$$\sup_g \{ \lambda_1^{inv}(g) \text{Vol}(g)^{2/n} \} = \infty,$$

where the metrics $g$ are $G$-invariant and conformal to $g_0$. 
Theorem [Korevaar]:

\[ \lambda_k(g) \operatorname{Vol}(g)^{2/n} \leq C_n([g_0]) k^{2/n}, \]

for any \( g \) conformal to \( g_0 \); \( C \) depends only on \( n \) and the conformal class \([g_0]\) of \( g_0 \).

Corollary Let \((M, g_0)\) and \( G \) be as on preceding slide. For any positive integer \( N \), there exists a \( G \)-invariant metric \( g_N \) conformal to \( g_0 \) such that none of the first \( N \) eigenfunctions of \( g_N \) is \( G \)-invariant.

Remark The assumption on the dimension of \( G \) is necessary in theorem on preceding slide. We can remove the conformal requirement and recover the same result for \( G \) a discrete group.
Restrictions: Invariance and Embedded

Let $g$ be an $O(n)$-invariant metric of volume 1 on $S^n$ embedded as a hypersurface of revolution in $\mathbb{R}^{n+1}$.

**Theorem [Colbois-D-El Soufi]:** For all $k$,

$$\lambda_k^{\text{inv}}(g) < \lambda_k^{\text{inv}}(D^n) \text{Vol}(D^n)^{2/n}.$$  

Furthermore, there exists a sequence $g_i$ of $O(n)$-invariant metrics on $S^n$ in $\mathbb{R}^{n+1}$ with

$$\lambda_k^{\text{inv}}(g_i) \text{Vol}(g_i)^{2/n} \to \lambda_k^{\text{inv}}(D^n) \text{Vol}(D^n)^{2/n},$$

but $\lambda_k^{\text{inv}}(D^n) \text{Vol}(D^n)^{2/n}$ is not attained by any smooth metric on $S^n$. 
Embedding is not enough!

**Proposition [Colbois-D-El Soufi]:** Within the class of smooth $S^1$-invariant metrics $g$ on $T^2$ which correspond to an embedding of $T^2$ in $\mathbb{R}^3$, 

$$\sup_g \{ \lambda_1^{\text{inv}}(g) \text{Vol}(g) \} = \infty.$$ 

**Remark** The argument also works for a general torus $T^{n+1} = S^1 \times S^n$. 


Future Directions

• Can we construct $G$-invariant metrics, $G$ discrete, such that $\lambda_1(g) \text{Vol}(g)^{2/n}$ gets arbitrarily large?

• What happens if we look at invariant $p$-forms, $p > 0$?

• For every $k \in \mathbb{N}$, there exists an integer $m(k, g) \geq k$ such that

$$\lambda_k^{\text{inv}}(g) = \lambda_{m(k, g)}(g).$$

What is the behavior of $m(k, g)$?