Listening to Orbifolds: What does the Laplace spectrum tell us?

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Slides available from
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The Plan:

1. Historical Motivation
2. Vibrating Strings
3. Drums and Manifolds
4. Orbifolds
Historical Motivation

Chemistry: identify elements by spectral “fingerprints”

Physics: development of quantum mechanics

Mathematics: how are knowledge of structure and knowledge of spectrum related?
Vibrating Strings

Setup: string of length $L$ with uniform density and tension, fixed endpoints

Pluck the string:

Describe motion of string with function $f(x, t)$

Constraints on $f(x, t)$:

- $f(0, t) = 0$
- $f(L, t) = 0$
- $\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2}$
Solving the Wave Equation

Look for waveforms, i.e. solutions $f(x,t)$ such that

$$f(x,t) = g(x)h(t)$$

$g(x)$ gives shape
$h(t)$ measures amplitude

Substitute solution into wave equation:

$$\frac{h''(t)}{h(t)} = \frac{g''(x)}{g(x)} = \lambda$$

so

$$T(h) = \lambda h, \quad T(g) = \lambda g$$
Waveforms

Specific waveforms oscillate at specific frequencies

Sound of string of length $L$ is overtone sequence

\[ \frac{1}{2L}, \frac{2}{2L}, \frac{3}{2L}, \ldots \]

Waveforms form basis for vector space of motion functions $f(x, t)$

Can calculate sound from length

Can “hear” the length of a string!
Can one hear the shape of a drum?

\[ D = \text{compact domain in Euclidean plane} \]

Describe motion with function \( f(x, y, t) \)

Constraints on \( f(x, y, t) \):  
- \( f(x_0, y_0, t) = 0 \) for all \((x_0, y_0)\) on boundary  
- \( -\frac{\partial^2 f}{\partial t^2} = -\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} := \Delta f \)
Waveforms Again

Sound of drum given by list of frequencies associated to waveforms

\[ f(x, y, t) = g(x, y)h(t) \]

Substitute solution into wave equation:

\[ \frac{\Delta g}{g} = -\frac{h''}{h} = \lambda \]

Frequencies of vibration = Eigenvalues of \( \Delta \) on \( D \)

Cannot explicitly calculate list of frequencies in general

Can hear area and perimeter of drumhead
You cannot hear the shape of a drum!
Manifolds

\[ M = \text{compact Riemannian manifold} \]
\[ \Delta = -div \ grad \]

Big Question: How much geometric information about \( M \) is encoded in the eigenvalue spectrum of \( \Delta \)?

Answers:

- dimension
- volume
- \( M = \) surface:
  Euler characteristic, hence genus
- round spheres characterized by spectra
- isospectral nonisometric Riemann surfaces
- isospectral nonisometric planar domains
What is an orbifold?

**Examples**

1. Manifolds

2. $M/\Gamma$, where $\Gamma$ is a group acting “nicely” on a manifold $M$

Let $M = S^2$, and let $\Gamma$ be the group of rotations of order 3 about the north-south axis. Then $M/\Gamma$ is a $(3, 3)$-football.
3. $\mathbb{Z}_p$-teardrop: topologically a 2-sphere, with a single cone point of order $p$
Riemannian Orbifolds

Construction of Riemannian metric on $O$:

- define metric locally via coordinate charts
- patch together
- must be invariant under local group actions

Define objects like function and Laplacian locally

Laplacian is well-behaved on orbifolds:

- $\text{Spec}(O) = 0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \cdots \uparrow \infty$
- Each eigenvalue $\lambda_i$ has finite multiplicity.
- Orthonormal basis of $L^2(O)$ composed of smooth eigenfunctions
Inverse Spectral Geometry of Orbifolds

• Gordon, Webb and Wolpert (1992): used orbifolds in construction of drum examples

• Gordon and Rossetti (2003): middle degree Hodge spectrum cannot distinguish Riemannian manifolds from Riemannian orbifolds

• Gordon, Greenwald, Webb and Zhu (2003): spectral invariant for footballs and teardrops

• Shams, Stanhope and Webb: there exist arbitrarily large (but always finite) isospectral sets, where each element in a given set has points of distinct isotropy
Listening to Orbifolds

\( O = \) compact Riemannian orbifold
\( \Delta = -\text{div grad} \) (locally)

Big Question: How much geometric information about \( O \) is encoded in the eigenvalue spectrum of \( \Delta \)?

Answers:

- dimension
- volume
- orbisurfaces: genus???
- isospectral nonisometric Riemann orbisurfaces???
Tools in Dimension 2

$O$: orbisurface with $s$ cone points of orders $m_1, \ldots, m_s$

Define the (orbifold) Euler characteristic of $O$ to be

$$\chi(O) = \chi(X_0) - \sum_{j=1}^{s} \left(1 - \frac{1}{m_j}\right).$$

**Theorem (Gauss-Bonnet)** Let $O$ be a two-dimensional Riemannian orbifold. Then

$$\int_O K dA = 2\pi \chi(O).$$

Euler characteristic is spectrally determined, but unknown if spectrum determines genus
Finiteness of Isospectral Sets

McKean showed that only finitely many compact Riemann surfaces have a given spectrum. We extend this result to the setting of Riemann orbisurfaces. Specifically, we show

**Theorem (D.)** Let $O$ be a compact Riemann orbisurface with genus $g \geq 1$. Then in the class of compact orientable hyperbolic orbifolds, there are only finitely many members which are isospectral to $O$. 
Huber’s Theorem for Compact Riemann Surfaces

**Theorem (Huber)** Two compact Riemann surfaces of genus \( g \geq 2 \) have the same spectrum of the Laplacian if and only if they have the same length spectrum.

length spectrum: sequence of all lengths of all oriented closed geodesics on the surface, arranged in ascending order
An Analog of Huber’s Theorem

**Theorem (D.-Strohmaier)** If two compact Riemann orbisurfaces are Laplace isospectral, then we can determine their length spectra and a sum involving the orders of the cone points. Knowledge of the length spectrum and the orders of the cone points determines the Laplace spectrum.
Selberg Trace Formula for Compact Riemann Orbisurfaces

\[ \sum_{n=0}^{\infty} h(r_n) = \frac{\mu(F)}{4\pi} \int_{-\infty}^{\infty} r h(r) \tanh(\pi r) dr \]

\[ + \sum_{\{R\} \text{ elliptic}} \frac{1}{2m(R) \sin \theta(R)} \int_{-\infty}^{\infty} \frac{e^{-2\theta(R)r}}{1 + e^{-2\pi r}} h(r) dr \]

\[ + \sum_{\{P\} \text{ hyperbolic}} \frac{\ln N(P_c)}{N(P)^{1/2} - N(P)^{-1/2}} g[\ln N(P)] \]
Sketch of Proof:

Use appropriate version of Selberg Trace Formula

- Plug in a “good” test function for $h(r)$
- Know volume from Weyl’s asymptotic formula
- Read off lengths of geodesics from singular support of wave trace
- Left with elliptic summand involving orders of cone points
Explicit Bounds

**Theorem (Buser)** Let $S$ be a compact Riemann surface of genus $g \geq 2$. At most $e^{720g^2}$ pairwise non-isometric compact Riemann surfaces are isospectral to $S$.

No $g$-independent upper bound is possible

Brooks, Gornet, and Gustafson examples: cardinality of set grows faster than polynomially in $g$
Bounds for Riemann Orbisurfaces

- Cubic pseudographs (D.)
- Fenchel-Nielsen parameters (D.)
- Collar theorem (D.-Parlier)
- Bers’ theorem (D.-Parlier)
- Understanding of geodesic behavior
Future Directions

- How do geodesics on orbifolds behave?
- For what classes of orbifolds are the isotropy types spectrally determined?
- What is the relationship between the spectrum of a Riemann orbisurface and that of the Riemann surface which finitely covers it?