Upper bounds for invariant eigenvalues of the Laplacian

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The Plan

1. Historical Motivation

2. Two Dimensions

3. Higher Dimensions:
   - $S^n$ with an action of $O(n)$
   - Manifolds with an action of a Lie group
Setup

M: compact connected Riemannian manifold of dimension $n \geq 2$

Laplacian: $\Delta = -\text{div grad}$

$\text{Spec}(g) = \{0 = \lambda_0(g) < \lambda_1(g) \leq \ldots \leq \lambda_k(g) \leq \ldots \}$

**Question** Consider the $k$th eigenvalue, normalised as a functional

$$g \rightarrow \lambda_k(g) \text{Vol}(g)^{2/n}$$

on the space of Riemannian metrics. What are critical/extremal metrics for this functional?
Answers for Surfaces

$S^2$ [Hersch] :

$$\lambda_1(g) \text{Vol}(g) \leq 8\pi = \lambda_1(g_{\text{can}}) \text{Vol}(g_{\text{can}})$$

Orientable surface of genus $\gamma$ [Yang-Yau]:

$$\lambda_1(g) \text{Vol}(g) \leq 8\pi \left[ \frac{\gamma + 3}{2} \right]$$

Surface of genus $\gamma$ [Korevaar]: There exists a universal constant $C > 0$ such that for all $k > 0$,

$$\lambda_k(g) \leq Ck(\gamma + 1).$$
Answers in Higher Dimensions

**Theorem [Colbois-Dodziuk]:**

\[
\sup_{g}\{\lambda_1(g) \text{Vol}(g)^{2/n}\} = \infty
\]

To get finite bounds, we need to add some restrictions...

- conformal class (Korevaar, El Soufi-Ilias)
- projective Kähler metrics (Bourguignon-Li-Yau)
- symplectic or Kähler metrics (L. Polterovich)
- invariance under isometries (Abreu-Freitas)
$S^1$ acts on $S^2$

Consider $S^2$ with metrics which are smooth, have total area $4\pi$, and are $S^1$-invariant.

Denote the invariant eigenvalues by $\lambda_{k}^{\text{inv}}(g)$.

**Theorem [Abreu-Freitas]:** In this setting, $\lambda_{1}^{\text{inv}}(g)$ can be any number strictly between 0 and $\infty$. 
Do more restrictions help get bounds?

Fixed Gauss curvature at poles: still have

\[ 0 < \lambda_1^{\text{inv}}(g) < \infty \]

Metrics embedded in \( \mathbb{R}^3 \):

\[ \lambda_k^{\text{inv}}(g) < \frac{1}{2} \xi_k^2 \]

and in particular

\[ \lambda_1^{\text{inv}}(g) < \frac{1}{2} \xi_1^2 \approx 2.89 \]

Can characterize the supremum geometrically
Questions

- How do the invariant eigenvalues of a manifold behave under general group actions?
- Is being embedded essential to bounding \( \lambda_k^{\text{inv}}(g)\text{Vol}(g) \)?
- If we find critical metrics, to what do they correspond geometrically?
Dimension 2

Nontrivial $S^1$-actions exist on

- sphere
- torus
- projective plane
- Klein bottle

**Proposition [Colbois-D-El Soufi]:** Within the class of smooth $S^1$-invariant metrics $g$ on $T^2$ which correspond to an embedding of $T^2$ in $\mathbb{R}^3$, 

$$\sup_g \{ \lambda_1^{\text{inv}}(g) \text{Vol}(g) \} = \infty.$$ 

**Remark** The argument also works for a general torus $T^{n+1} = S^1 \times S^n$. 


Higher Dimensions

Consider $O(n)$-invariant metrics on $S^n$ embedded in $\mathbb{R}^{n+1}$, of volume 1.

**Theorem [Colbois-D-El Soufi]:** For all $k$,

$$\lambda_{k}^{\text{inv}}(g) < \lambda_{k}^{\text{inv}}(D^n).$$

Furthermore, there exists a sequence of embeddings of $S^n$ in $\mathbb{R}^{n+1}$ with

$$\lambda_{k}^{\text{inv}}(g_i) \to \lambda_{k}^{\text{inv}}(D^n),$$

but $\lambda_{k}^{\text{inv}}(D^n)$ is not attained by any smooth metric on $S^n$. 
Lie Groups

**Assumptions**

- \( \dim(M) \geq 3 \)
- \( G: \) Lie group of dimension \( \geq 1 \) acting on \( M \) by isometries
- \( \dim(M/G) = d \geq 1 \)

**Theorem** [Colbois-D-El Soufi]: Let \((M, g_0)\) and \(G\) be as above. Then

\[
\sup_{g} \left\{ \lambda_1^{\text{inv}}(g) \frac{\text{Vol}(g)^{2/n}}{\text{Vol}(g_0)^{2/n}} \right\} = \infty,
\]

where the metrics \( g \) are \( G \)-invariant and conformal to \( g_0 \).
Discrete Lie Groups

**Theorem** [Korevaar]: Let $(M, g_0)$ be a compact Riemannian manifold and $G < \text{Isom}(M)$ a discrete group such that the quotient $M/G$ is orientable. Then

$$\sup_{g} \{ \lambda_k^{\text{inv}}(g) \text{Vol}(g)^{2/n} \} \leq C_n(g_0)k^{2/n},$$

where $g$ is a $G$-invariant metric conformal to $g_0$, and $C$ depends only on $n$ and $g_0$. 
Removing the conformal requirement

**Theorem [Colbois-D-El Soufi]:** Let \((M, g_0)\) be a compact Riemannian manifold of dimension \(n \geq 3\), and \(G < \text{Isom}(M)\) a discrete group. Then

\[
\sup_{g} \{ \lambda_1^{\text{inv}}(g) \text{Vol}(g)^{2/n} \} = \infty.
\]
Idea of Proof
Future Directions

- Can we construct $G$-invariant metrics, $G$ discrete, such that $\lambda_1(g)\text{Vol}(g)^{2/n}$ gets arbitrarily large?

- What happens if we look at invariant $p$-forms, $p > 0$?

- Can we say more about critical invariant metrics?