Inverse Spectral Problems on Riemannian Orbifolds

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What is an orbifold?

**Examples**

1. Let $\Gamma$ be a group acting properly discontinuously on a manifold $M$ with fixed point set of codimension 2 or greater. Then the quotient space $M/\Gamma$ is an orbifold.

2. $\mathbb{Z}_p$-teardrop: topologically a 2-sphere, with a single cone point of order $p$
EXAMPLE Orbifolds arising from triangle groups: topologically a 2-sphere, with three cone points
Points in $O$ with nontrivial isotropy groups are called *singular points*, and the collection of all such singular points is the *singular set* $\Sigma_O$.

**Example** Manifolds are orbifolds for which the singular set is empty.
Why are orbifolds of interest?

1. Visual way to understand group acting on a space
2. Easiest singular spaces
3. Crystallography
4. String theory
5. Study of 3-manifolds
Riemannian Orbifolds

Construct Riemannian metric on $O$ by defining metrics locally via coordinate charts and patching metrics together using a partition of unity.

Structures must be invariant under local group actions.

Results of local analysis hold, but global results may not hold or take new form.

Every point $p$ in a Riemannian orbifold has a fundamental coordinate chart.
**Definition** Let $O$ be a compact Riemannian orbifold. A map $f : O \to \mathbb{R}$ is a **smooth function** on $O$ if for every coordinate chart $(U, \tilde{U}/\Gamma, \pi)$ on $O$, the lifted function $\tilde{f} = f \circ \pi$ is a smooth function on $\tilde{U}$.

If $O$ is a compact Riemannian orbifold and $f$ is a smooth function on $O$, then we define the Laplacian $\Delta f$ of $f$ by lifting $f$ to local covers. That is, we lift $f$ to $\tilde{f} = f \circ \pi$ via a coordinate chart $(U, \tilde{U}/\Gamma, \pi)$. We denote the $\Gamma$-invariant metric on $\tilde{U}$ by $g_{ij}$ and set $\rho = \sqrt{\det(g_{ij})}$. Then we can define

$$\Delta \tilde{f} = \frac{1}{\rho} \sum_{i,j=1}^{n} \frac{\partial}{\partial \tilde{x}^i} (g^{ij} \frac{\partial f}{\partial \tilde{x}^j} \rho).$$
**Theorem** (Chiang) Let $O$ be a compact Riemannian orbifold.

1. The set of eigenvalues $\lambda$ in $\Delta f = \lambda f$ consists of an infinite sequence $0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \cdots \uparrow \infty$. We call this sequence the spectrum of the Laplacian on $O$, denoted $\text{Spec}(O)$.

2. Each eigenvalue $\lambda_i$ has finite multiplicity.

3. There exists an orthonormal basis of $L^2(O)$ composed of smooth eigenfunctions $\phi_1, \phi_2, \phi_3, \ldots$, where $\Delta \phi_i = \lambda_i \phi_i$. 
1966: Kac asked “Can one hear the shape of a drum?”

Examples: Milnor, Vignéras, Sunada’s method, submersion method

2000: Dianu showed that every indexed one-pointed torus is uniquely determined up to isometry by the first few lengths in its length spectrum

2002: Gordon and Rossetti showed that the middle degree Hodge spectrum cannot distinguish Riemannian manifolds from Riemannian orbifolds

2003: Gordon, Greenwald, Webb, Zhu calculated the first few invariants of the heat expansion for bad orbifolds
**Definition** Let $O$ be a 2-orbifold with $r$ corner reflectors of orders $n_1, \ldots, n_r$ and $s$ cone points of orders $m_1, \ldots, m_s$. Then we define the (orbifold) Euler characteristic of $O$ to be

$$
\chi(O) = \chi(X_0) - \frac{1}{2} \sum_{i=1}^{r} (1 - \frac{1}{n_i}) - \sum_{j=1}^{s} (1 - \frac{1}{m_j}),
$$

where $\chi(X_O)$ is the Euler characteristic of the underlying space of $O$.

**Theorem (Gauss-Bonnet)** Let $O$ be a two-dimensional Riemannian orbifold. Then

$$
\int_{O} K dA = 2\pi \chi(O),
$$

where $K$ is the curvature and $\chi(O)$ is the orbifold Euler characteristic.
**Theorem (Farsi)** Let $O$ be a closed orientable smooth Riemannian orbifold with eigenvalue spectrum $0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \ldots \uparrow \infty$. Then for the function $N(\lambda) = \sum_{\lambda_j \leq \lambda} 1$ we have

$$N(\lambda) \sim (\text{Vol } B_0^n(1))(\text{Vol } O) \frac{\lambda^{n/2}}{(2\pi)^n}$$

as $\lambda \uparrow \infty$. Here $B_0^n(1)$ denotes the $n$-dimensional unit ball in Euclidean space.

Consequences:

1. Laplace spectrum determines an orbifold’s dimension and volume
2. Dimension 2: spectrum determines an orbifold’s Euler characteristic
Proposition Fix $g \geq 0$ and $m \geq 2$. Let $O$ be an orientable hyperbolic 2-orbifold of genus $g$ with exactly one cone point of order $m$. Let $O'$ be in the class of orientable hyperbolic 2-orbifolds of genus $g$ with cone points of orders 2 and higher, and suppose that $O$ is isospectral to $O'$. Then $O'$ must have exactly one cone point, and its order is also $m$. 
Proof  Let $O$ and $O'$ be orientable hyperbolic 2-orbifolds with the same genus, i.e. $\chi(X_O) = \chi(X_{O'})$. Further suppose that $O$ is isospectral to $O'$. Then $\chi(O) = \chi(O')$.

Suppose that $O'$ has one cone point of order $n_1$. It follows that

$$\frac{1}{m} = \frac{1}{n_1},$$

or $m = n_1$.

Now suppose that $O'$ has two cone points of orders $n_1$ and $n_2$. Then

$$\frac{1}{m} + 1 = \frac{1}{n_1} + \frac{1}{n_2}.$$

But $n_i \geq 2$ for $i = 1, 2$, so $\frac{1}{n_1} + \frac{1}{n_2} \leq 1$. This is a contradiction, hence $O$ and $O'$ are not isospectral. This argument is easily extended to $k > 2$ cone points of orders $n_1, \ldots, n_k$.  \(\square\)
We can extend this proposition to the case of two orbifolds with different underlying spaces.

**Proposition** Let $O$ be an orientable hyperbolic 2-orbifold of genus $g_0 \geq 0$ with $k$ cone points of orders $m_1, \ldots, m_k$, where $m_i \geq 2$ for $i = 1, \ldots, k$. Let $O'$ be an orientable hyperbolic 2-orbifold of genus $g_1 \geq g_0$ with $l$ cone points of orders $n_1, \ldots, n_l$, where $n_j \geq 2$ for $j = 1, \ldots, l$. Let $h = 2(g_0 - g_1)$. If $l \geq 2(k + h)$, then $O$ is not isospectral to $O'$.

**Corollary** Fix $g \geq 0$. Let $O$ be an orientable hyperbolic 2-orbifold of genus $g$ with $k$ cone points of orders $m_1, \ldots, m_k$, $m_i \geq 2$ for $i = 1, \ldots, k$. Let $O'$ be an orientable hyperbolic 2-orbifold of genus $g$ with $l \geq 2k$ cone points of orders $n_1, \ldots, n_l$, $n_j \geq 2$ for $j = 1, \ldots, l$. Then $O$ is not isospectral to $O'$. 
McKean showed that only finitely many compact Riemann surfaces have a given spectrum. We extend this result to the setting of orbifold Riemann surfaces. Specifically, we show

**Theorem** Let $O$ be a compact hyperbolic orientable 2-orbifold with genus $g \geq 1$ and cone points of order three and higher. Then in the class of compact hyperbolic orientable orbifolds, there are only finitely many members which are isospectral to $O$. 
Future Directions:

1. Explicit bounds on the size of isospectral sets
2. Examples of large families
3. Understand orbifold injectivity radius
4. What properties of orbifolds are spectrally determined?