Optimal Product Bundling with Dependent Valuations: the Price of Independence

Mihai Banciu
School of Management,
Bucknell University
Lewisburg, PA, 17837
mmb018@bucknell.edu

Fredrik Ødegaard
Ivey Business School,
Western University
London, ON N6A 3K7
fodegaard@ivey.uwo.ca

May 17, 2016

Abstract

In this paper we investigate the tactical problem of pricing a bundle of products when the underlying valuations of the bundle components are dependent. We use copula theory to model the joint density of reservation prices and provide analytical derivations for the prices under different bundling strategies and sharp bounds for the profit function. We discover that when only the bundle is offered and the marginal costs are relatively small, the seller is better off by bundling products that have a negative association between their valuations, while the converse is true when the marginal costs are relatively high. We also show that the net benefit of offering a full product line containing both the bundle and the components decreases for mild to strong associations between the component valuations, compared to offering just the bundle. Finally, we analyze how the typical literature assumption of independence of reservation prices impacts the seller’s profitability when in fact the valuations are dependent, and find that this gap in profitability, which we call the “price of independence”, can be arbitrarily large.

Keywords: dependence; pricing; bundling; revenue management; copulas
1 Introduction

Bundling, i.e. selling a package of individual products or services as one unit, is an extensively used marketing strategy in many industries. For example, online travel aggregators such as Expedia or Travelocity provide vacation packages consisting of hotel, airfare, car rental, and tickets to local attractions; content providers such as Condé Nast and HBO offer multi-platform access to their products, such as printed and digital, or TV and streaming; computer and mobile phone manufacturers pre-load hardware with various software packages; and telecommunication companies provide “triple-” or “quadruple-play” packages which include landline phone, broadband Internet, digital television, and wireless phone services. Bundling is widely used in practice because it acts essentially as a price-discrimination mechanism [Stigler, 1963], it reduces the buyers’ heterogeneity [Schmalensee, 1984] and thus enables the seller to extract additional consumer surplus, as long as the seller has market power. Even when competition effects are present, a telecommunication company such as Verizon can leverage its quadruple-play package and make it very hard for a regional telecommunications company to compete in smaller markets. In this case, bundling is virtually a deterrent strategy for an established firm and acts as an entry barrier [Nalebuff, 2004].

In general, companies usually choose one of three possible bundling strategies [Adams and Yellen, 1976]. Under Pure Components (PC) the seller chooses to sell only the components, but not the bundle, while under Pure Bundling (PB) she offers only the bundle for sale, but not the components. Finally, under Mixed Bundling (MB) she offers the bundle(s) as well as the components, separately, for sale. When in fact all possible combinations of bundles and separate components are offered, we say that the seller uses a full mixed bundling strategy and when only a subset of all the possibilities ends up being offered, then she uses a partial mixed bundling strategy.

One interesting—and generally hard—problem in bundling is how to price (optimally) the various bundles that the seller can offer. The difficulty of the problem stems from the different dimensions that can be considered in formulating an acceptable answer. For example, first consider the number of components, $N$, that can be bundled. Clearly, the full mixed bundling strategy is the most challenging to implement since the number of total packages (i.e. bundles plus components) that need to be priced equals $2^N - 1$. In contrast, the PC strategy requires pricing only the $N$ components, while the PB strategy involves the pricing of a single package. In addition to the exponential number
of decision variables, the mixed bundling strategy also involve an exponential number of pricing constraints, since a typical bundle price needs to be offered at a lower price than the sum of its component prices (otherwise the buyers can assemble the bundle for themselves). For these reasons, the majority of the bundling literature tends, mainly for tractability, to focus on $N = 2$ products. There are, however, notable exceptions. Hanson and Martin [1990] use a mathematical programming approach to compute optimal bundle prices as well as identifying which bundles should be offered, assuming that all reservation prices can be correctly identified. Bitran and Ferrer [2007] use a multinomial choice model to estimate what bundles should be offered, considering those offered by the competition. Ibragimov and Walden [2010] consider the problem of pricing a bundle consisting of a finite number of components with heavy tailed valuations, while Ferrer et al. [2010] considers dynamic pricing of a line of interchangeable bundles consisting of a product and service.

The second difficulty that appears in bundle pricing is due to the possible dependence structure between consumers’ valuations for the components. Additionally, consumers’ valuation function for the bundle can be additive, sub-additive, or super-additive in the components, thus reflecting differently the substitutability/complementarity relationship of the components. For example, a conglomerate such as General Electric manufactures and sells under its Appliances division various household appliances such as washers and dryers, for which customers may have divergent preferences. A retailer such as H&M frequently bundles different clothing items together using intuitive heuristic pricing rules such as “buy one product and get the second for 50% off” (thus, enabling the consumer to combine items that could be either complements, such as a shirt and a pair of pants, or substitutes, such as two shirts of different colors.) Similarly, in a quadruple-play bundle, a fixed telephone landline and a mobile phone plan serve intrinsically the same basic communication need, but the flexibility afforded by each of these options is quite different. Hence, depending on their particular situations, consumers searching for a phone contract or to furnish their newly-purchased home could have either a positive or negative association in their valuation of the products.

Most of the extant bundle pricing literature [e.g. Venkatesh and Mahajan, 1993, Eckalbar, 2010, Bhargava, 2013, among others], primarily for analytical tractability reasons, has focused strictly on the analysis when goods have independent consumer valuations. Several notable exceptions are the papers of Venkatesh and Kamakura [2003] who look explicitly at pricing bundles of complements and substitutes (although their analytical results are limited to the pure bundling/pure components scenarios), Banciu et al. [2010]
who examine the bundling of substitutes in a vertical market, Armstrong [2013], who examines the profitability of bundling with sub- and super-additive component valuations, and McCardle et al. [2007] who extend the analysis of independent valuations to the cases of either perfectly positive and negative correlated components (although their results are limited to uniform marginal distributions). Interesting papers that address bundling with dependent valuations are Schmalensee [1984] who examines the profitability of bundling under correlated valuations described by the Normal distribution and Chen and Riordan [2013], who explicitly model the dependence component structure using copula functions while examining the general question of bundle profitability.

The main research aim of this paper is to investigate a combination of the first two dimensions discussed above. Specifically, we focus on two things. First, we look at how firms should price the individual products/services as well as the bundle, when the demands for the components exhibit structural dependence under each of the aforementioned bundling strategies. This is important for a lot of industries where bundling is prevalent, such as traditional retail or e-commerce. In an online setting, finding good quality solutions is critical if the bundles are created on the fly—for example, travel aggregators such as Expedia or Orbitz who offer travel packages need to price these bundles in real time, while the results page is loading for the consumers visiting the site. Second, we examine what happens when, either for convenience or due to poor marketing research efforts, bundles are priced as if there were no dependence relationship among the components. Therefore, in order to make the analysis more tractable and to keep our results comparable with the existing literature, we will make the following assumptions throughout the paper: we limit the analysis to a single seller (monopoly) offering two products ($N = 2$); we assume that consumers have bundle valuations that are additive in the components.

A shortcoming of (incorrectly) assuming independent valuations is that it may lead to mis-specified models [Jedidi et al., 2003], and thus resulting in a forfeit of revenue. We show that although the association between the consumers’ valuations of the bundle components may be rather weak (possibly even appearing as independent), correctly accounting for dependence (in either direction) can provide significant incremental revenue. Furthermore, modeling the relationship between the products via a generic dependence structure allows us to partially capture some of the substitutability/complementarity effects that have typically been modeled using a sub- or super-additive bundle valuation function [Eppen et al., 1991, Venkatesh and Kamakura, 2003]. The main benefit of our approach is that, in general, it is easier to estimate the magnitude of these effects in practice, since sellers
can usually measure the consumers’ valuations for the components via marketing research tools, but measuring directly the bundle valuation for all possible bundles is not as easy a task.

In order to accommodate the dependence structure between the components, we draw on the framework of copula functions. A copula function represents a statistical construct that “couples” the two component distributions and synthetically creates the joint valuation distribution, from which one can derive the bundle valuation via convolution. This proposed approach is particularly attractive, since one can create many different copula functions, for any choice of component pairs. The flexibility of this approach becomes clear if we make the simple observation that constructing a joint density with different valuations is very difficult using traditional methods (e.g. the valuation for a product follows a Lognormal distribution, while the valuation for the second product may be Gamma distributed). Moreover, in a multivariate setting, there may be a lot of partial correlations that are impossible to capture analytically; the copula approach bypasses these difficulties and provides tractability.

While copula functions have been discovered since the 1940s, in the general business literature they have been relatively slow to percolate. A notable exception is finance where copula functions have been heavily used, in particular in the theory of quantitative risk management [Kakouris and Rustem, 2014], where modeling the default risk of portfolios of correlated assets is of utmost importance. Some other isolated exceptions include Clemen and Reilly [1999] (a decision analysis paper), and for our purposes the more relevant papers of Meade and Islam [2010], who model the time between inter-purchases, Danaher and Smith [2011], a comprehensive survey about the the applications of copula theory in marketing, and Chen and Riordan [2013], who, to the best of our knowledge, were the first to propose studying bundling with dependent valuations using copulas. In particular, our basic model setup (a monopolist selling two products and a bundle) and choice of copula functions are consistent with Chen and Riordan [2013]. However, our research goals diverge significantly from theirs. The main thrust of Chen and Riordan [2013] is to extend the seminal work of McAfee et al. [1989] by establishing sufficient conditions which guarantee the (weakly) dominance of the mixed bundling strategy in terms of profitability, under dependence. Hence, their work is concerned with the behavior of the seller’s revenue function under a mixed bundling strategy. In contrast, we aim for determining in closed-form the expressions for the optimal prices, so that we can investigate the influence of their respective drivers. In the process, we also extend the results in Chen and Riordan
2 Modeling Dependence Using Copulas

Suppose that a seller considers offering two products and/or, potentially a bundle consisting of the two products (hereinafter referred to as components). Each component has marginal cost of production \( m_i \), is sold at a price \( p_i \geq m_i \), \( i = 1, 2 \), and the marginal cost of assembling the bundle is \( m = m_1 + m_2 \) (thus, we do not capture effects due to production economies) and the monopolist prices the bundle at \( p_b \). In general, if a bundle is offered, then it is “survivable” when \( m \leq p_b \leq p_1 + p_2 \). We assume that a consumer’s valuation \( X_i \) for each component \( i \) is a continuous random variable with distribution function \( F_i(x) = \Pr(X_i \leq x) \), defined on the bounded support \([0, a_i]\), \( i = 1, 2; a_i < \infty \). The extension to distribution functions with unbounded support is straightforward but generally limited to those with finite first and second moments. Demand for each component is either zero or one unit. Furthermore, if components 1 and 2 form a bundle, then the bundle valuation is the random variable \( X_b = X_1 + X_2 \) with realization \( x_b = x_1 + x_2 \). A consumer’s purchasing rule is surplus maximizing, that is, she will buy product \( i \) whenever \( x_i - p_i > \max\{x_j - p_j, x_b - p_1 - p_2, 0\} \), \( i, j = 1, 2, i \neq j \), and both products (separately or as a bundle) whenever i) \( x_1 + x_2 - p_1 - p_2 > \max\{x_1 - p_1, x_2 - p_2, 0\} \), (under PC), or ii) \( x_b - p_b > 0 \) (under PB), or iii) \( x_b - p_b > \max\{x_1 - p_1, x_2 - p_2, 0\} \) (under MB); otherwise there is no purchase.

We explicitly assume that the joint distribution of consumers valuations \( F(x_1, x_2) = \Pr(X_1 \leq x_1, X_2 \leq x_2) \) with support on \([0, a_1] \times [0, a_2] \) is not known and that it allows for dependence, i.e. \( F(x_1, x_2) \) may not necessarily equal \( F_1(x_1)F_2(x_2) \). We only require the seller to have knowledge regarding the sign and magnitude of the strength of the association between the two components; we capture this information by the parameter \( \theta \). Then, based on the marginal distributions \( F_1(x_1), F_2(x_2) \), and \( \theta \), the seller can either estimate the joint distribution, denoted by \( \hat{F}(\cdot) \), or assume the joint distribution \( F(\cdot) \) by way of a so-called

\[ \text{In other words, we are not considering applications where marginal valuations follow, say, power law distributions with heavy tails. Nevertheless, to gain insight into dependence and bundle pricing under long tailed valuations in Section 4.3 we conducted a simulation study based on Weibull and Pareto distributions.} \]
A two-dimensional copula (copula, for short) is a function $C : [0, 1]^2 \rightarrow [0, 1]$ with the following properties:

1. For any $u, v \in [0, 1], C(u, 0) = C(0, v) = 0$ and $C(u, 1) = u, C(1, v) = v$;

2. For all $u_1, u_2, v_1, v_2 \in [0, 1]$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$,

$$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0.$$  

The first property is commonly referred to as the grounded and uniform marginals property and it implies that a) if any marginal outcome is 0, then the joint probability of all the outcomes is also equal to 0, and b) that the copula marginals are uniformly distributed. The second property is called the 2-increasing property and it is equivalent to the well known rectangle inequality associated with any bivariate distribution. Although copula modeling can be extended beyond two dimension, for the purpose of this paper we restrict the discussion to two-dimensional copulas. For more background and in depth discussion regarding copula modeling, see Joe [1997] or Nelsen [2006].

Notice that the domain of the copula is the unit square, implying that one can potentially use distribution functions as arguments to the copula function. The following result, due to Sklar, formally establishes this relationship; see Nelsen [2006] for the proof.

**Sklar’s Theorem.** Let $X_1$ and $X_2$ be univariate random variables with distribution functions $F_1$ and $F_2$, respectively, and joint distribution function $F$. Then, there exists a copula $C$ such that $F(x_1, x_2) = C(F_1(x_1), F_2(x_2))$. Conversely, if $C$ is a copula and $F_1$ and $F_2$ are distribution functions, then the function $F$ induced by $C$ is a joint distribution function with margins $F_1$ and $F_2$.

According to Sklar’s Theorem, not only can one build joint continuous distributions using copulas, the actual copula is unique given the choice of marginals, in other words, $C(x_1, x_2) = F(F_1^{-1}(x_1), F_2^{-1}(x_2))$, where $F^{-1}(\cdot)$ indicates the pseudo-inverse (quantile) of the distribution function. In the statistics literature, researchers have identified several copula families that can be used to model a variety of phenomena (cf. Nelsen [2006], Joe [1997]). For example, a well-known copula is the product copula, defined as $C_{\Pi}(u, v) = uv$, which simply captures the independence relationship between the marginals. Three
other well-studied copulas of interest to us due to the possibility of allowing both negative and positive dependencies, as well as symmetry and asymmetry, are the Farlie-Gumbel-Morgenstern (FGM) copula, for \( \theta \in [-1, 1] \),

\[
C_{\text{FGM}}(u, v) = uv[1 + \theta(1 - u)(1 - v)],
\]

(1)

and the so-called family of Archimedean copulas, which includes (among others) the Frank copula, for \( \theta \in \mathbb{R} \setminus \{0\} \),

\[
C_{\text{F}}(u, v) = \frac{-1}{\theta} \ln \left( 1 + \frac{(e^{-\theta u} - 1)(e^{-\theta v} - 1)}{e^{-\theta} - 1} \right),
\]

(2)

and the Gumbel-Hougaard copula, for \( \theta \in [1, \infty) \),

\[
C_{\text{G}}(u, v) = \exp \left\{ -\left[ (-\ln u)^\theta + (-\ln v)^\theta \right]^{1/\theta} \right\}.
\]

(3)

In all cases the parameter \( \theta \) models the strength of the relationship between the two components: when \( \theta < 0 \) or \( \theta > 0 \) then there is either a negative or a positive association between the components, respectively. (Note that this interpretation is valid only for the copula families for which, e.g., \( \theta > 0 \) captures positive dependence; the paper’s conclusions would be reversed if the converse parametrization were true.) Independence is similarly captured by select values of \( \theta \): \( C_{\text{FGM}} = C_{\Pi} \) when \( \theta = 0 \), \( C_{\text{G}} = C_{\Pi} \) when \( \theta = 1 \) and \( C_{\text{F}} = C_{\Pi} \) when \( \theta \to 0 \). The bundle valuation model and the joint distribution associated with the bundle can be naturally extended to accommodate an arbitrary number of components.

For a given continuous and twice differentiable (almost everywhere) copula \( C(\cdot) \), define the partial derivatives \( C_1(u, v) = \frac{\partial C}{\partial u} \), \( C_2(u, v) = \frac{\partial C}{\partial v} \) (the conditional marginals) and \( c(u, v) = \frac{\partial^2 C}{\partial u \partial v} \) (the copula density). It follows from Sklar’s Theorem and by differentiation that the consumers’ valuations joint density function is given by

\[
f(x_1, x_2) = f_1(x_1)f_2(x_2)c(F_1(x_1), F_2(x_2)).
\]

The key insight here is that the joint density is modeled by capturing two effects: the product of the individual marginals (as under independence), multiplied by the copula density (which captures the dependence and acts as a weighting factor to ensure the overall integration of the space to 1). Figure 1 depicts the pure and mixed bundling strategies regions for an FGM copula with Uniform\([0, 1]\) marginals when \( \theta = -.5 \) and \( \theta = .75 \).
Due to the copula properties given in the definition, one can transform the original valuation space where each consumer is characterized by the valuation pair \((x_1, x_2) \in [0, a_1] \times [0, a_2]\) to one where the valuation pair is \((F_1(x_1), F_2(x_2)) \equiv (u, v) \in [0, 1]^2\), bound by the joint copula density \(c\). This suggests two possible avenues for handling the dependency between the components: either solve everything in the original joint distribution space, or obtain an intermediate solution in terms of the cumulative distribution function in the copula space, and then recover the optimal prices by applying the quantile function, i.e. find optimal valuation \(u^*\) and let \(p_1^* = F_1^{-1}(u^*)\). The second approach works because of the bijective relationship between the distribution and the quantile functions and, moreover, this property allows one to use a particular instantiation of the copula function (e.g. \(C_{FGM}\) or \(C_F\)) as a translating mechanism between the two spaces. Figure 2 below illustrates both approaches, assuming that \(X_1 \sim \text{Exp}(0.75)\) truncated on \([0, 2]\) and \(X_2 \sim \text{Exp}(0.5)\) truncated on \([0, 5]\), bound by a FGM copula function with \(\theta = 0.75\) under a pure components strategy.

Notice that the nature of the dependence relationship captured by the copula function need not be linear and therefore we need a different measure that generalizes the commonly used Pearson linear correlation coefficient \(\rho\). Two such concordance measures or rank correlations are Kendall’s \(\tau\) and Spearman’s \(\rho_s\). For brevity we restrict our at-
tention to Kendall’s \( \tau \), which for any copula \( C \) is defined [Joe, 1997, pg.32] as follows
\[ \tau = 4 \int_0^1 \int_0^1 C(u, v) \, dC(u, v) - 1. \]
It is easy to see that if \( C = C_{\Pi} \) then \( \tau = 0 \), otherwise for any other copula choice \(-1 \leq \tau \leq 1\). It should be noted there is a one to one correspondence between the copula parameter and the associated Kendall’s \( \tau \). In particular, for the FGM copula, \( \tau_{FGM} = 2\theta/9 \), for the Frank copula, \( \tau_F = 1 + 4[D_1(\theta) - 1]/\theta \), where \( D_1(\theta) = \theta^{-1} \int_0^\theta t(e^t - 1)^{-1} \, dt \) is a Debye function of the first kind, while for the Gumbel copula \( \tau_G = 1 - 1/\theta \). This suggests that the FGM copula, while attractive analytically, can only capture relatively weak dependencies between the valuations (i.e. \( \tau \in [-2/9, 2/9] \)), the Frank copula can capture an arbitrary dependency level (i.e. \( \tau \in [-1, 1] \)), and the Gumbel copula can capture the entire spectrum of positive association (i.e. \( \tau \in [0, 1] \)), with the latter two typically sacrificing analytical tractability. In Appendix A we discuss how to estimate \( \tau \) and consequently \( \theta \) from a sample.
3 Bundle Pricing

In this section we characterize each of the three possible bundling strategies. We first provide a cursory examination of the pure components (unbundled) strategy, and then focus our analysis on pure and mixed bundling, respectively. Without loss of generality we normalize $a_1 = 1$ and assume $1 \leq a_2 < \infty$, so that we can capture in our analysis both the partial as well as the full mixed bundling strategies. For readability, we define $a_2 \equiv a$, and provide all proofs in Appendix B.

3.1 Pure Components (Unbundled Sales) Analysis

Under this strategy, the seller does not offer the bundle. If the consumers are interested in both products, they need to assemble the bundle themselves, by purchasing both components separately. The market shares associated with each component are, for $i = 1, 2$,

$$Q_i = \Pr(X_i > p_i) = 1 - C(F_i(p_i), 1) = 1 - F_i(p_i),$$

and the seller profit is $\Pi_{PC} = (p_1 - m_1)Q_1 + (p_2 - m_2)Q_2$. Interestingly, these quantities turn out to be identical to those derived under the assumption of independent valuations. Consequently, under the PC strategy there will be no difference between the optimal component prices (and thus overall profit), regardless of whether the independence assumption holds or not. However, since the unbundled scenario serves as a baseline case in most of the literature when comparing profitability across different bundling strategies [e.g. McAfee et al., 1989], we formalize our finding in Proposition 1 below. First, define by $h_i(x) = f_i(x) / [1 - F_i(x)]$ the hazard rate and by $g_i(x) = xh_i(x)$ the generalized hazard rate associated with the random variable $X_i$, $i = 1, 2$. Since we are pricing a multi-product line we make the assumption that $h_i(x)$ is weakly increasing (i.e. $X_i$ has Increasing Failure Rate or is IFR), such that we maintain closure under addition and ensure unimodality of the profit function [Banciu and Mirchandani, 2013].

**Proposition 1.** Under the PC strategy, the optimal prices $p_i^*$ satisfy $(p_i - m_i)h_i(p_i) = 1$, $i = 1, 2$. Moreover, if $m_i = 0$, then the optimal price for component $i$ is $p_i^* = g_i^{-1}(1)$, where $g_i^{-1}(x)$ is the inverse of $g(x)$.

---

2Extensions to unbounded support with finite first and second moments are straightforward, while extensions to, for example, marginal power law distributions with exponent less than 1 would require further analysis.
Unlike the independence case, the fraction of buyers who end up buying both products is, in fact, influenced by the underlying dependence structure. The size of this segment is simply the joint survival distribution \( \bar{C}(u, v) = \Pr(X_1 \geq u, X_2 \geq v) = 1 - u - v + C(u, v) \), that is, \( Q_{12} = \bar{C}(F_1(p_1), F_2(p_2)) \). The contribution of \( Q_{12} \) to the seller’s profit is \( \tilde{\pi} = (p_1 + p_2 - m_1 - m_2)Q_{12} \), which is easily seen to have the same behavior as \( Q_{12} \), i.e. increasing whenever the valuations are positively dependent and decreasing otherwise. For example, assuming a FGM copula with uniform individual valuations, one can verify that \( \frac{\partial \tilde{\pi}}{\partial \theta} = p_1 p_2 (1 - p_1)(1 - p_2)(p_1 + p_2 - m_1 - m_2) \geq 0 \). The intuition behind the identity is that as the positive dependence between the components’ valuations increases, the fraction of buyers purchasing both products tends to increase. Since the prices are not influenced by the underlying relationship between the components, \textit{caeteris paribus}, an increase in demand which is not accompanied by an increase in prices will always result in a higher profit.

### 3.2 Pure Bundling Analysis

Under this strategy, the seller never offers the separate components. If the consumers are interested in only one of the components, they must purchase the bundle and discard the other component. Note that the two marginal distributions do not necessarily have to be identical on the same bounded support, so \( F_1(x) \neq F_2(x) \), for at least some \( x \in (0, 1) \). The market demand \( Q_b = \Pr(X_1 + X_2 > p_b) \) for the bundle is given by

\[
Q_b = \begin{cases} 
1 - \int_0^{F_1(p_b)} \int_0^{F_2(p_b)[1-u/F_1(p_b)]} c(u, v) \, dv \, du & \text{if } 0 \leq p_b \leq 1, \\
1 + F_2(p_b - 1) - \int_0^1 \int_{F_2(p_b-1)}^{\beta u} c(u, v) \, dv \, du & \text{if } 1 < p_b \leq a, \\
\int_0^{1-F_1(p_b-a)} \int_{F_2(p_b-1)-\gamma(1+u)}^1 c(u, v) \, dv \, du & \text{if } a < p_b \leq a + 1
\end{cases}
\]

where \( \beta = F_2(p_b) - F_2(p_b - 1) \) and \( \gamma = \frac{1-F_2(p_b-1)}{1-F_1(p_b-a)} \). The resulting total profit under the PB strategy is then \( \Pi_{PB} = (p_b - m)Q_b \).

The following lemma establishes the necessary condition that the optimal bundle price must satisfy assuming a pure bundling strategy.

**Lemma 2.** The optimal bundle price under a pure bundling strategy satisfies

\[
p_b^* = m - \left( \frac{d}{dp_b} \ln Q_b \right)^{-1}
\]
In general, equation (5) is hard to provide in closed-form without a specific copula choice and marginal distribution. Suppose, without loss of generality (since asymmetrical valuations do not drive the pure bundling strategy), that the maximum valuation for both products is $a = 1$ and therefore $F(x_1, x_2) : [0, 1]^2 \rightarrow [0, 1]$. For tractability, throughout the rest of the section we use the FGM copula to capture the dependence (although in section 4 we will also provide numerical examples based on the Frank and Gumbel copulas), and assume that component valuations are uniformly distributed, that is, $F_1(x) = F_2(x) = F(x) = x$, for all $x \in [0, 1]$. This distributional assumption simply states that the demand function for each separate component is linear, which is consistent with the extant literature. Furthermore, by focusing on the uniform distribution we are able to derive closed form solutions regarding the near-optimal pricing strategies and provide a more intuitive discussion. However, we stress that the proposed model formulation is general and can accommodate any distributional assumption. We can now rewrite (4) as

$$Q_b = \begin{cases} 
1 - \int_0^{p_b} C_1(u, p_b - u) \, du & \text{if } 0 \leq p_b \leq 1 \\
2 - p_b - \int_{p_b - 1}^1 C_1(u, p_b - u) \, du & \text{if } 1 < p_b \leq 2 
\end{cases}$$

(6)

where for the FGM copula $C_1(x, y) = y[1 + \theta(1 - 2x)(1 - y)]$.

We start with a well-known result from the bundling literature [Venkatesh and Kamakura, 2003] which will serve as a baseline. Let $C = C_{II} = uv$, i.e. the two components are independent. Then, $Q_b = 1 - p_b^2 + \int_0^{p_b} (p_b - x) \, dx = 1 - p_b^2/2$, which yields $p_b^* = (m + \sqrt{m^2 + 6})/3$ whenever $m \leq 1/2$, and $Q_b = 2 - p_b - \int_{p_b - 1}^1 (p_b - x) \, dx = (2 - p_b)^2/2$, leading to $p_b^* = 2(m + 1)/3$ otherwise. If in particular $m = 0$, then the optimal price is the well-known value $p_b^* = \sqrt{2}/3$.

The following proposition presents the optimal bundle price if the joint density can be modeled with a FGM copula.

**Proposition 3.** If the components valuations are uniformly distributed in $[0, 1]$ and bound by the FGM copula $C_{FGM}$, the optimal pure bundling price is given by the corresponding non-negative solution to the following quartic equations:

a) for $0 \leq p_b \leq 1$,

$$5\theta p_b^4 - 4\theta(4m + 4)p_b^3 + 3[3(\theta + 1) + 4\theta m]p_b^2 - 6m(\theta + 1)p_b - 6 = 0$$
b) for $1 < p_b \leq 2$,

$$5\theta p_b^4 - 4\theta (m + 4)p_b^3 + 3[3(\theta + 1) + 4\theta m]p_b^2 - 2[3m(\theta + 1) + (4 - \theta)]p_b + 3(4m + 3) = 0.$$  

While the equations introduced in Proposition 3 can be solved directly and expressed algebraically, the closed form expression is very involved. In order to present a more intuitive expression which also allows comparative statics, we used one single iteration of Newton’s method to approximate the solution to the quartic equations, using $(m + \sqrt{m^2 + 6})/3$ and $2(m + 1)/3$, respectively, as the starting points for each case (i.e. the analytical solutions under independent valuations). The results are formally presented in the following Proposition, where the main takeaway is that the near-optimal prices are nonlinear in $\theta$.

**Proposition 4.** When the joint valuation is modeled by the FGM copula and the individual valuations are uniformly distributed in $[0, 1]$, the near-optimal bundle price $\tilde{p}_b^*$ is given by

a) for $0 \leq \tilde{p}_b^* \leq 1$,

$$\tilde{p}_b^* = \frac{m + \eta}{3} + \frac{\theta}{3} \left[ \frac{4m^4 - 4m^3(3 - \eta) - 12m^2(1 + \eta) + 12m(9 - \eta) + 9(16\eta - 37)}{81\eta - 9\eta[432 - 36m(2 - \eta) + 4m^2(9 - \eta) - 4m^3 - 141\eta]} \right],$$  

(7)

where $\eta = \sqrt{m^2 + 6}$;

b) for $1 < \tilde{p}_b^* \leq 2$,

$$\tilde{p}_b^* = \frac{2(m + 1)}{3} + \frac{8\theta}{3} \left[ \frac{(2 - m)^2(m + 1)}{81 + \theta(8m^2 + 40m - 31)} \right].$$  

(8)

Note that one needs to impose additional restrictions on the $(m, \theta)$ pair in order to ensure that each price is a proper solution; for example, when $\theta = 0$, the restrictions are $m \leq 1/2$ and $m > 1/2$. for part a) and part b), respectively. One important application of Proposition 4 is that of pricing bundles of digital or information goods, where typically $m = 0$. In that situation, we obtain the following closed form solution, presented in Corollary 5.
Figure 3: Optimal (dots) and near-optimal (line) prices and revenues under pure bundling vs. unbundled sales (dashed).

Corollary 5. When \( m = 0 \), the near-optimal bundle price \( \tilde{p}_b^* \) is given by

\[
\tilde{p}_b^* = \sqrt{\frac{2}{3}} + \frac{\theta}{\sqrt{6}} \left[ \frac{16\sqrt{6} - 37}{27 - \theta(24\sqrt{6} - 47)} \right].
\]

In order to evaluate the quality of the approximation, we performed a computational experiment where for a large (100 \( \times \) 100) selection of feasible \((\theta, m)\) pairs we computed both the optimal bundle price and the approximate optimal bundle price. See Figure 3, where the left column panels show the optimal bundle price and the right column panels show the resulting profit \( \Pi_{PB} \) as a function of the dependence parameter \( \theta \). In all cases
the single iteration Newton’s method performed remarkably well, giving an average root mean squared error of less than 0.5% both for the prices and for the revenue. We note that the near-optimal bundle price has the properties that we would expect, in that it is increasing both in $m$ and in $\theta$. This can also be confirmed by analyzing equations (7) and (8). While the intuition behind the behavior with respect to the variable cost $m$ is obvious, we can explain the effect of $\theta$ by noting that as the customers tend to put higher (lower) valuations on both bundle components, then the seller can exploit this behavior by raising (lowering) her prices. With highly negative valuations, and thus with customers preferring one product, but not the other, a bundle discount would convince customers who are “on the margin” to self-select into purchasing the bundle. Hence, in this situation as the bundle price increases, then the demand for the bundle would decrease and the overall bundle premium that the seller extracts would decrease; and vice versa. Thus, we would expect the overall profit to decrease in $\theta$. This is indeed the outcome whenever the marginal cost is relatively small ($m \leq 1/2$). This happens because as the prices increase, more customers would opt out from purchasing the bundle, and this drop in demand happens at a faster rate than the rate at which the prices increase.

On the other hand, when the marginal costs are relatively large ($m > 1/2$), we observe, counter-intuitively, that the profit is actually increasing when the valuations are positively dependent ($\theta > 0$ for this copula). We can explain this by noticing that in this case the seller is able to “dig deeper” in the customer base whenever the component association is positive and therefore, even though the prices still increase in $\theta$, the demand will fall at a slower pace and thus the overall profit will be increasing. In this case, the bundle demand appears to be inelastic. The interesting managerial implication of this observation is that the seller can raise the profitability of her pure bundling strategy if she either bundles components with negative associations whenever the marginal costs are small, or components with positive associations whenever the marginal costs are high. This is what we sometimes observe in practice. For example, DirecTV bundles tens of cable channels in a single package and customers tend to have negative associations between many pairs of channels (say, Food Network and Golf Channel), but the marginal cost of producing and delivering this bundle to their subscribers is negligible compared to the revenue collected. Conversely, GE Appliances is often running promotions where they bundle a washer and a dryer, for which presumably the marginal cost of production is relatively large. In this case, a household either has the durable goods and will not value the bundle highly (unless they consider updating), or it does not, in which case the components would have large
valuations, thus implying a positive association between the goods. Conversely, if the customers only own either the washing machine or the dryer, then they will have little interest in purchasing the bundle, and therefore the seller does not profit as much, even if the bundle price is relatively low.

Finally, copula theory provides a powerful tool for bounding the value of the profit function. Frank et al. [1987] find and express *sharp* bounds for the distribution function of $Z = X_1 + X_2$. In Proposition 6 we use their main result to find lower and upper bounds for the profit function under the pure bundling strategy, while maintaining our original assumption of uniform component valuations. Naturally, the result can easily be extended to other marginal densities.

**Proposition 6.** Suppose that the component valuations are symmetrical and uniformly distributed. Then, for any copula $C$ that connects the components to form the bundle valuation, the profit function $\Pi_{PB}$ satisfies the following inequalities:

\[
(p_b - m)(1 - p_b) \leq \Pi_{PB} \leq (p_b - m) \quad \text{if } 0 \leq p_b \leq 1 \quad (9)
\]

\[
0 \leq \Pi_{PB} \leq (p_b - m)(2 - p_b) \quad \text{if } 1 < p_b \leq 2. \quad (10)
\]

Furthermore, these bounds are sharp.

The implication of Proposition 6 is that the profit under pure bundling can be bounded from above and from below regardless of the copula choice, as long as the component valuations are known. For example, we know that under independence, i.e. $C = C_{\Pi}$, the optimal bundling price under zero marginal costs is $p_b = \sqrt{2}/3$. For this value, Proposition 6 gives the bounds $0.15 \leq \Pi_{PB} \leq 0.816$ (in this case we know the true optimal profit to be 0.544). We illustrate these bounds for the case of zero marginal cost in Figure 4. The shaded region indicates the feasible region of the profit function for any copula, while the dotted line indicates the profit function for the independence copula $C_{\Pi}$. Notice that the zone which maximizes the feasible profit range occurs at $p_b = 1$, and this may explain why sometimes firms may simply price the bundle at the sum of optimal prices under pure components and offer no bundle discount: lacking any solid knowledge about the bundle demand, it is intuitive for the seller to instinctively try to maximize her opportunities of collecting a higher revenue. Our analysis supports the claim that it is always better to offer a moderate discount, since it drives the price into a region where the lower bound on the profit is positive, and thus the risk associated with actually collecting a very small
revenue is reduced. Indeed, it would appear that, lacking any means of estimating the magnitude of the dependence relationship, a seller can effectively implement a maximin pricing strategy, whereas by choosing a suitable small price (in this case, the bundle price would be $p_b = 0.5$) she is guaranteed a non-negative revenue; thus, she is maximizing her minimum revenue!

### 3.3 Mixed Bundling Analysis

Under this strategy, the seller offers both the components separately and as a bundle. We consider both full mixed bundling, where components 1 and 2, and the bundle of components 1 and 2 are sold, and partial mixed bundling, where component 1 and the bundle of components 1 and 2 are sold. Consequently, in the latter setting consumers that are interested in component 2 only, must buy the bundle and discard component 1. For the mixed bundling scenario we assume the joint consumer valuation is given by $F(x_1, x_2) : [0, 1] \times [0, a] \rightarrow [0, 1]$, where $a \geq 1$.

For the full mixed bundling strategy the total market share $Q_1$ for product 1 is given by

$$Q_1 = \int_{0}^{F_2(p_b - p_1)} \int_{F_1(p_1)}^{1} c(x_1, x_2) \, dx_1 \, dx_2 = F_2(p_b - p_1) - C(F_1(p_1), F_2(p_b - p_1)),$$
Similarly, the total market share for product 2 is \( Q_2 = F_1(p_b - p_2) - C(F_1(p_b - p_2), F_2(p_2)) \). The market share for the bundle customers is

\[
Q_b = \int_{F_1(p_1)}^{1} \int_{F_2(p_1)}^{1} c(x_1, x_2) \, dx_2 \, dx_1 - \int_{F_1(p_1)}^{F_1(p_2)} \int_{F_2(p_1)}^{F_2(p_2)} c(x_1, x_2) \, dx_2 \, dx_1
\]

\[
= 1 - F_1(p_b - p_2) - F_2(p_b - p_1) + C(F_1(p_1), F_2(p_b - p_1))
\]

\[
- \int_{F_1(p_1)}^{F_1(p_2)} C_1(x_1, F_2(p_b - F_1^{-1}(x_1))) \, dx_1,
\]

and the firm’s profit function is \( \Pi_{MB} = (p_1 - m_1)Q_1 + (p_2 - m_2)Q_2 + (p_b - m)Q_b \). Conversely, in the partial mixed bundling scenario, the market share for component 1 and bundle, respectively, are

\[
Q_1 = F_2(p_b - p_1) - C(p_1, p_b - p_1)
\]

\[
Q_b = 1 - F_2(p_b - p_1) - \int_{F_1(p_1)}^{1} C_1(x, F_2(p_b - F_1^{-1}(x))) \, dx
\]

Just like in the PB scenario, we maintain the assumption of an underlying FGM copula, and we are interested in determining the optimal prices for the components and for the bundle. However, in this MB scenario there are two possible dominating strategies, each driven by the underlying product valuation. If the maximum valuation for the second product is relatively small (i.e. \( 1 \leq a < 2 \)), then the full mixed bundling strategy is dominant; hence, in this scenario \( a \) is simply a normalization constant and without loss of generality we can assume it to be 1. Conversely, if the maximum valuation for the second product is relatively high (i.e. \( a \geq 2 \)) then the seller is better off by only offering product 2 as part of the bundle and never separately, in order to avoid potential cannibalization issues—therefore, a partial mixed bundling strategy dominates in this case. With this caveat in place, in order to compute our near-optimal prices, we will use a one step gradient descent approximation method, since it produces an intuitive solution without compromising the quality. As starting points, we use the optimal mixed bundling prices computed under independence, that is \( p_1 = p_2 = 2/3, p_b = (4 - \sqrt{2})/3 \) for the full case, and \( p_1 = 2/3, p_2 = p_b = (2 + 3a)/6 \) for the partial MB case [Bhargava, 2013], with the gradient of the profit function defined as \( \nabla \Pi_{MB} = (\partial \Pi_{MB}/\partial p_1, \partial \Pi_{MB}/\partial p_2, \partial \Pi_{MB}/\partial p_b)^T \).

We summarize the result below in Proposition 7 and provide a numerical illustration in Figure 5.
Proposition 7. Under the mixed bundling strategy, with dependent uniform valuations bound by the FGM copula and moderate marginal prices \((0 \leq m_1, m_2 \leq 0.466)\), the near-optimal prices are given by:

a) for the full spectrum mixed bundling strategy \((a = 1)\):

\[
\tilde{p}_1^* = \frac{2}{3} + \frac{2 - \sqrt{2} - \theta}{3} \left[ 12 - 10\sqrt{2} m_1 - \sqrt{2} m_1 + (4\sqrt{2} - 2)m_2 \right],
\]

\[
\tilde{p}_2^* = \frac{2}{3} + \frac{2 - \sqrt{2} - \theta}{3} \left[ 12 - 10\sqrt{2} m_2 - \sqrt{2} m_2 + (4\sqrt{2} - 2)m_1 \right],
\]

\[
\tilde{p}_b^* = \frac{4 - \sqrt{2}}{3} + \frac{(1 + \sqrt{2})(m_1 + m_2)}{3} + \frac{\theta}{27} \left[ \frac{28\sqrt{2} - 17}{9} - \frac{(6 + 5\sqrt{2})(m_1 + m_2)}{3} \right].
\]

b) for the partial spectrum mixed bundling strategy \((a \geq 2)\):

\[
\tilde{p}_1^* = \frac{2}{3} + \frac{m_1(6 - \theta)}{12} - \frac{m_1 + m_2}{3a} + \frac{\theta}{27a^2} \left[ m_1 + 4m_2 - \frac{2(3a - 2)}{3} \right],
\]

\[
\tilde{p}_b^* = \frac{2}{3a} + \frac{m_1 + 3m_2}{3a} + \frac{2\theta}{81a^2} \left[ \frac{15a - 4}{3} - 2(2m_1 + 5m_2) \right].
\]

Besides providing a close approximation for the profit function, the price specifications from Proposition 7 are useful for comparative statics, as well as for providing additional intuition about bundling behavior. First, note that the component prices are linear and decreasing in \(\theta\), while the bundle price is linear and increasing in \(\theta\); see also Figure 5 and contrast with the results for the PB strategy. As we have shown in Section 3.2, bundling is profitable when the component valuations are negatively dependent and this is what we see in the mixed bundling setting. As the component valuations shift from negative to independence to positive association, the seller will lower her component prices, since the customer mass is shifting either into the bundle quadrant or into the no purchase region, thus causing a drop in the demand for individual components. On the other hand the subsequent increase in demand for the bundle clearly entices the seller to raise her bundle prices. However, since the rate at which the customer leave the individual segment is higher than the rate at which they enter the bundle segment, the net profit effect, as evidenced in Figure 5b, is an overall decrease in \(\theta\).
3.4 The revenue gap between mixed and pure bundling

The second interesting takeaway of the analysis from the previous part concerns the relationship between the profitability of mixed and pure bundling due to the shape of the underlying components valuation distribution. To investigate this we modeled the valuation pairs using Beta distributions with identical specifications for each component, because of their versatility. We connect the valuations with a Frank copula, in order to capture a large range of dependency, spanning both the negative and positive association spectrum. The four Beta distributions used are Beta(1,1) (i.e. uniform distribution), Beta(1,2), Beta(2,1) (i.e. linear densities and , with equal variance but different skewness), and Beta(2,2) (i.e. a concave, paraboloid-shaped distribution , with smaller variance than the Uniform). Then, for all values of between -10 and 20 in increments of 0.1 we computed the relative revenue gap between the mixed and pure bundling strategies; we set \( m_1 = m_2 = 0 \). We summarize the results in Figure 6.

Interestingly, in three cases, the gap disappears for extreme values of \( \theta \), that is, there does not appear to be any added benefit in terms of profitability from mixed bundling. Previous studies have established the weak superiority of (mixed) bundling in terms of profitability, irrespective of whether the consumer valuations are independent [McAfee et al., 1989, Fang and Norman, 2005] or not [Schmalensee, 1984, Chen and Riordan, 2013], as long
as inventory is unlimited and the products are horizontally differentiated. Clearly, our finding is relevant given the algorithmic challenge of mixed bundling, since it implies that when the bundle components are highly dependent in either direction, pure bundling appears to be asymptotically optimal (as long as the component valuations are not negatively skewed) and therefore the bundle pricing problem reduces to just finding one optimal price. The fourth case shows that mixed bundling exhibits higher relative profitability under negative dependency. This is also intuitive, because the Beta(1,2) distribution models a population of “high-spending” customers, with high valuations for the bundle. While previous work [Schmalensee, 1984, e.g.] has examined the influence of customer heterogeneity (as captured by the variance), on mixed bundling profitability, our results show the importance of considering skewness, since both linear densities have equal variance but the Beta(2,1) distribution, which captures the “thrifty” consumers exhibits no benefit from mixed bundling under mild association. This observation generalizes the results in Venkatesh and Kamakura [2003, Result 4], who examine numerically only a scenario consisting of uniform marginals, by showing not only that the net benefit of mixed bundling can vanish even for moderate associations between components, but also that mixed bundling can be more profitable even for strong negative associations.

Finally, Bitran and Ferrer [2007] argued that bundling works because it can capture customers that have relatively moderate valuations for both products. What we observe in our results is a different kind of moderation, specifically, that mixed bundling tends to work best when there exists a moderate association relationship between the component
valuations (specifically, a mild positive association), as well as limited skewness in each valuation distribution. In particular, since the FGM copula only captures a limited range of dependencies, we have a posteriori empirical validation for using its analytical specification throughout this paper.

4 The Price of Independence

As mentioned in the introduction, one of the more common assumptions in the bundling literature is that of independence among bundle components’ valuations. If this belief does not hold in reality, then not only is the model misspecified, but the seller can effectively leave “money on the table” since she will set her prices at a suboptimal level. Within a given bundling strategy, we refer to the relative revenue loss as the price of independence (PoI), formally defined as

$$\text{PoI} = \frac{\Pi(p^*)}{\Pi(p)} - 1,$$

where $\Pi(p)$ is the profit under the corresponding bundling scenario (PB or MB) and price vector $p = (p_1, p_2, p_b)$, $p^*$ is the vector of optimal prices given the true dependence structure, and $p_I$ is the vector of (suboptimal) prices assuming independence between the components valuation. Recall that in the PB scenario there is only a bundle price $p_b$ that the seller needs to set ($p_1, p_2$ are not defined), while in the full spectrum MB scenario the seller has to set components prices $p_1, p_2$ and bundle price $p_b$.

Note that the optimal price vector $p^*$ also depends on the particular copula and dependence parameter $\theta$. We write $p(\theta)$ to represent the vector of prices for a given $\theta$. For the FGM and Frank copulas the independence assumption translates to the seller (incorrectly) assuming $\theta = 0$. For other copulas it would have different implications (e.g. the Gumbel family of copulas captures independence when $\theta = 1$). In order to reduce confusion, all figures in this section have Kendall’s $\tau$ on the horizontal axis, since unambiguously independence is captured when $\tau = 0$ (recall that for each copula there is a one-to-one mapping between $\theta$ and $\tau$).

The objective of this section is to quantify and analyze the expected PoI for FGM, Frank and Gumbel copulas. We first analyze the theoretical PoI values, under both PB and MB scenarios, when the seller has perfect knowledge of the copula and dependence parameter $\theta$ i.e. when $p^* = p^*(\theta)$. In Section 4.2, we perform a robustness check to evaluate the PoI under the MB scenario, when prices are based on an estimated dependence parameter $\hat{\theta}$,
i.e. when $p^\ast = p^\ast(\hat{\theta})$. We also analyze the consequences on PoI when prices are based on an estimated dependence parameter $\theta$ from an incorrect copula function, i.e. when the seller sets prices assuming, say, a FGM copula when in fact demand can be adequately captured by a Frank copula (and vice-versa).

4.1 The Magnitude of the Price of Independence

We are interested in studying what happens to the PoI, in particular as $\theta$ approaches some extreme value given the actual copula function which best captures the joint valuations. For example, if the actual specification is $C_{FGM}$, under the PB strategy with uniform valuations, we find, using straight numerical nonlinear optimization, that the worst upper bound for PoI is 4.17%, hence for $C_{FGM}$, $0 \leq \text{PoI} \leq 4.17\%$, with the upper bound attained when $m = 1/2, \theta = 1$. To extend our analysis for both pure and mixed bundling strategies we varied, for $m \in [0, 1]$ (pure bundling) and $m_1, m_2 \in [0, 1]$ (mixed bundling) in increments of 0.1 units, computed the corresponding PoI and then averaged the PoI across these instances for $-1 \leq \theta \leq 1$, or equivalently, $-0.22 \leq \tau \leq 0.22$. In order to get a better understanding of the PoI when the dependence relationship is relatively strong, we also repeated the analysis for both Frank and Gumbel copulas, since one is radially symmetrical but the other is not; in this case we expanded the range of $\theta$ to $-10 \leq \theta \leq 10$, or equivalently, $-0.66 \leq \tau \leq 0.66$ for Frank and $\theta$ to $1 \leq \theta \leq 20$, or equivalently, $0 \leq \tau \leq 0.95$ for Gumbel. For each such scenario corresponding to a particular choice of a $(m, \theta)$ pair or $(m_1, m_2, \theta)$ triplet, respectively we used numerical optimization to obtain the locally optimal solution under dependence which we then used as inputs to compute the profits assuming independence. The results of the numerical analysis are presented in Figure 7.

First, notice that regardless of the copula specification used, for a given magnitude of dependence, the (average) price of independence under pure bundling appears to be higher than under mixed bundling, but the PoI graphs exhibit the same curvature. While this difference is not as pronounced under the FGM and Gumbel copulas, in the Frank case the difference is about an order of magnitude smaller—notice that in Figure 7c the left y-axis measures the PoI under pure bundling (ranging from 0% to 90%), while the right y-axis measures the PoI under mixed bundling (ranging from 0% to 8%). Intuitively, this can be explained by the better segmentation achieved under mixed bundling, in particular the possibility of “divesting” from the bundle toward the individual components, as the mass of consumers shifts. This shift causes an increase in component prices and a decrease in bundle
Figure 7: Expected PoI under FGM, Frank, and Gumbel copulas, for PB (solid) and MB (dashed) strategies.

prices, but the overall revenue remains somewhat stable, thus diminishing the magnitude of the PoI. Furthermore, the component prices are less sensitive to the dependence structure and therefore, in general, the mixed bundling scenario will have a lower PoI. To see the intuition, consider the pure bundling scenario where component prices are unaffected by the dependence structure.

Second, the parabola from Figure 7a suggests that if the dependence relationship is captured by a different copula specification, which could allow for stronger relationships (remember that the FGM copula can only model relatively weaker dependencies), the PoI can grow arbitrarily large. Indeed, a different pattern emerges if we model the joint density...
using the Frank copula $C_F$, which allows for the entire spectrum of positive and negative associations, in a symmetrical fashion. The PoI is decidedly asymmetrical, with the gap increasing for negative values of $\theta$, while for positive values of $\theta$ the PoI appears to be finite (see Figure 7c). Clearly, what we are observing here is that, under a negative association between the components (which geometrically will cluster along the northwestern-southeastern diagonal of the valuation space), the seller is able to better segment the market and extract additional profits from offering the bundle. Conversely, when the valuations are strongly positively associated (and thus clustered along the southwestern-northeastern diagonal of the valuation space), due to the symmetric nature of the copula, a high mass of customers will have very small valuations for the bundle and thus self-select into the “no purchasing” segment. In this case, the opportunity loss captured by the PoI is clearly limited by the smaller concentration of bundle buyers. Similarly, for the Gumbel copula (which exhibits non-trivial upper tail dependence), the upper bound for PoI is approximately 10% attained when the bundle components exhibit perfect positive dependence. In contrast with the Frank copula, which can also capture arbitrarily large negative dependencies, the Gumbel copula captures only positive dependencies, that is, bundles where the components can function more or less like substitutes, thus reducing the overall profitability of any bundling strategy. However, even in this case, the magnitude of the PoI is non-trivial. Interestingly, under the mixed bundling scenario, the PoI assuming a Gumbel copula does not appear to be monotonic, suggesting that perhaps, highly dependent components dilute somewhat the mistaken independence assumption, since bundling benefits generally decrease as the strength of the positive relationship between components increases.

The above discussion can be generalized by considering the bounds in Proposition 6. For instance, for zero marginal cost, and assuming a non-degenerate copula with unimodal profit function, the upper bound on PoI for pure bundle is 300%. In fact, for the pure bundling scenario and positive marginal cost, it can easily be seen that the PoI can be made arbitrarily large; by extension the same result would apply to the mixed bundling strategy as well. The intuition is supported by Figure 4: since the lower bound of the profit function under PB is zero and attainable for some copula, then the PoI can get arbitrarily bad for some price $p_b \geq 1$, which is a function of the copula parameter $\theta$. For instance, one can consider an asymmetrical copula, which can have arbitrary mass concentrated in the northeast region of the valuation space. An interesting though rather technical analysis that we leave for future studies is to investigate how the bounds of PoI can be characterized for a given copula. Another interesting issue would be to further investigate the
characterization of the *optimal* mixed bundling strategy for a given copula and marginal distribution(s), e.g. under what conditions is the optimal mixed bundling strategy degenerating into a pure components versus pure bundling strategy. For instance, Ibragimov and Walden [2010] show that under a particular dependence structure with heavy tail marginal distributions, a pure component strategy dominates pure bundling. Nevertheless, the managerial implication of our observations are clear: sacrificing dependency for the sake of convenience or tractability may lead to a large amount of foregone profit.

4.2 Simulation-based Robustness Analysis: The Effects of Misspecification

Next we investigate the price of independence when the seller’s decision is based on an estimate of the dependence parameter. Our main interest is to evaluate the price of independence for the full mixed bundle scenario when the dependence parameter is unknown and has to be estimated. To perform this analysis we conduct three simulation studies for three different copulas: FGM, Frank, and Gumbel. In all cases we assume the marginal distributions are Uniform(0,1). The main purpose with the robustness analysis is to see how sensitive the theoretical PoI results are to estimates of the dependence parameter.

The simulation protocol is as follows. First, we randomly generate 100 pairs of observations for a given value of $\theta$ and specific copula-marginal distribution combination. Second, based on an estimated Kendall’s $\tau$ from the 100 pairs we determine $\hat{\theta}$ according to the specific copula (see discussion at end of Section 2). Third, based on the estimated $\hat{\theta}$ we numerically search for the locally optimal set of component prices $p_1$, $p_2$ and bundle price $p_b$. Finally, we randomly generate 100 data-sets each consisting of 100 pairs of observations, and based on the identified prices from the previous step and on the fixed prices assuming independence, we calculate the average PoI over the 100 data-sets. We repeat the simulation 500 times for 11-12 different values of the true underlying $\theta$: for the FGM copula $\theta = 0, \pm .2, \pm .4, \pm .6, \pm .8$, corresponding to $\tau = 0, \pm .04, \pm .09, \pm .13, \pm .18, \pm .22$; for the Frank copula $\theta = \pm .5, \pm .2, \pm .4, \pm .6, \pm .8, \pm 10$, corresponding to $\tau = \pm .06, \pm .21, \pm .39, \pm .51, \pm .6, \pm .66$; and finally, for the Gumbel copula $\theta = 1, 1.5, 2, 2.5, 3, 3.5, 4, 5, 6, 7, 8$, corresponding to $\tau = 0, .33, .5, .67, .71, .75, .8, .83, .86, .88$.

The results from our simulation study are summarized in Figure 8. The left and right side displays box-plots of the optimal prices and average PoI, respectively, over the 500 simulation instances where marginal distributions are Uniform(0,1). In Figure 8, the left
Figure 8: Simulation results of optimal prices and PoI under Uniform marginal distribution and FGM (left), Frank (middle) and Gumbel (right) copulas. Note, in (b) the bundle prices for the Frank copula with $\tau = -.66, -.60, -.51, -.39$ are not included.
box-plots include for baseline comparison the horizontal line representing the fixed price under independence; for the component prices, the solid line is at 0.67, while for the bundle price the dashed line is at 0.86. In panel (c) of Figure 8 the four cases corresponding to the strongest negative dependence have been omitted in order to avoid overlapping box-plots.

A first observation to make is how close the simulation-based results are to the analytical prices presented in Figure 5, and the expected PoI illustrated in Figure 7. In Figure 8, we observe that the median component prices to be decreasing in \( \tau \) while the median bundle price is increasing in \( \tau \), and a similar parabola shape for the median value of the average PoI. Furthermore, in the left graphs for \( \tau = 0 \) (independence) the median simulation-based optimal prices are very close to the true optimal values. It is noteworthy to highlight that the distribution in values shown in Figures 8 comes from that the estimated dependence parameter \( \hat{\theta} \) based on the original 100 randomly drawn pairs of observations. Consequently, occasionally over the 500 simulations, an ‘extreme’ dataset is generated, resulting in an ‘inaccurate’ estimate of \( \theta \), and hence the general performance is no better and no worse than simply assuming independence between the two components. We see this in particular for \( \tau \) values close to zero. On the other hand for even modest magnitudes of the underlying \( \tau \), the PoI interquartile range is strictly above zero. In other words, even with uncertainty around estimating \( \theta \) there is a significant value in adjusting the prices to accommodate for the dependence structure.

The main goal of our simulation study is to analyze the effect on PoI due to potentially misspecifying the dependence relationship. We have therefore kept the analysis based on perfect knowledge regarding both the marginal distributions and the specific copula function. This would, naturally, rarely be the case. In reality, sellers would likely have to estimate both the marginal valuation distributions and determine the appropriate copula. Note though, that some of the additional misspecification would affect both the “independent” and the “dependent” pricing strategy equally. For instance, based on the IFM estimation procedure (see Appendix A) the marginal distributions are fitted first, and hence both pricing strategies would be based on the same potentially misspecified marginal distribution. On the other hand, based on the insight from Figure 4 it is not clear what might be more ‘costly’: using an incorrect copula function or incorrectly using the independence copula in deriving ‘optimal’ prices.

To gain some insight regarding PoI when an incorrect copula is used we modified the above simulation study to analyze two cases: (1) when the underlying data is generated from a Frank copula, but incorrectly estimated by FGM copula; and (2) when the under-
(a) Uniform Marginals Assuming FGM when Frank (b) Uniform Marginals Assuming Frank when Gumbel.

Figure 9: Simulation results for PoI with marginals connected by (I) Frank copula but estimated as FGM (left), and (II) Gumbel copula but estimated as Frank (right).

lying data is generated from a Gumbel copula, but incorrectly estimated by Frank copula. In other words, we followed the exact same protocol as above with the exception that data is generated from a different copula than what is used to estimate the dependence parameter and to determine the corresponding “optimal” prices (we have kept the underlying marginal distributions unchanged and assumed to have been estimated correctly, in order to isolate only the effects due to the wrong choice of copula family). The results of the study are shown in Figure 9. In panel (a) the box-plot represent the PoI when the underlying data is generated from a Frank copula but estimated by FGM, while panel (b) correspond to the case when the underlying data is generated from a Gumbel copula but estimated by Frank. In both cases the marginal valuation distributions are drawn from Uniform(0,1).

The first thing to observe is that the general trends in Figure 9 are consistent with those shown in Figures 7 and 8 but at a lower level. In other words, we see that even if the underlying copula is misspecified there is still a significant benefit in adjusting prices to account for dependence although the percentage gains are not at the same levels as with
perfect information. The second thing to observe is that the shape of the curves in Figure 9 are consistent with those in Figure 7, implying that the main driver of the PoI is the true underlying dependence relationship between the components. For instance, when the true underlying copula is the Frank we observe an asymmetric PoI where the values for positive dependence tail off, while when the true underlying copula is Gumbel the PoI is non-monotone and tapers off around $\tau = .7$. Although our robustness analysis regarding misspecifying the copula is limited to the two cases, it provides empirical support to the claim that ignoring the effects of dependence can lead to a substantial loss in revenue.

4.3 Simulation-based Robustness Analysis: The Effects of Marginal Distributions with Long or Heavy Tails

The focus of our earlier analysis and simulation study thus far has been on cases where the marginal distributions have bounded support (or at least finite first moment). In this section we are interested to explore the effect on PoI when the marginal distributions have long and heavy tails. To perform this analysis we conduct two simulation studies for the FGM copula and two different marginal distributions: Pareto(1,1) and Weibull(.5,1).
Both of these marginal distributions are long tailed since their shape parameter are less than or equal to 1 (we use 1 and .5 for Pareto and Weibull, respectively); additionally, the Pareto(1,1) distribution is heavy tailed since it satisfies $\lim_{x \to \infty} F(x)/x^{-1} = 1$. A heavy tailed distribution would typically have a large mass of customers with relatively low component valuations and small but non-negligible mass of customers with extremely high component valuations. This leads to some interesting bundling situations; see Ibragimov and Walden [2010] for a study of bundling under heavy-tailed valuations. All studies assume zero marginal costs and symmetry, in that both marginal distribution samples are drawn from the same distribution, i.e., either both valuations are Weibull or both are Pareto. The simulation protocol is exactly as specified above with the exception that the marginal distributions follow a Pareto or Weibull distribution rather than Uniform.

The results from our simulation study are summarized in Figure 10. The left and right panels display box-plots of the average PoI for the Pareto and Weibull distributions, respectively, over the 500 simulation instances. There are a couple of interesting observations to note. First, we again see the similar parabola shape of the PoI. Second, comparing the two graphs with graph (b) in Figure 8, we see that the PoI values are roughly at par with each other with values between 0 and 2%. Intuitively these results are explained by that, although the Pareto and Weibull distributions have heavy tails and that the effect of dependence may diminish in the tail [Ibragimov et al., 2015], the optimal bundle and component prices are outside of those tails, i.e. where there is predominant mass and where dependence has a greater effect. Therefore, despite that our simulation study was only limited to the Pareto(1,1) and Weibull(.5,1) distributions, we find that our earlier PoI results seem robust with regard to marginal distributions with long and heavy tails.  

5 Conclusions

Pricing is an important tactical marketing problem. With markets becoming more competitive and more complex, correct marketing and pricing strategies are today even more important, particularly those involving bundling. Today we see firms not only producing traditional bundles by lumping together separate individual products or services, but also firms dismantling complete services which are then piecewise reassembled into add-ons or bundles. A notable example of the latter is the airline industry where many companies have

---

3We extended the simulation study to also include Pareto and Weibull marginals with the Frank and Gumbel copulas. The results were consistent with those shown in Figure 8.
started to bundle at a price services that used to be complimentary, like early boarding, preferred seating, in-flight entertainment, meals, and (of course) checked baggage service. Another example includes the recent “unbundling” policies of cable-TV packages implemented by the Canadian Radio-television and Telecommunications Commission (CRTC). By March 2016, Canadian cable-TV firms will only be permitted to offer a basic cable-TV bundle for $25 and then customers can build their own mix-&-match bundles [Bradshaw, 2015].

However, in this area, a major practical and theoretical bundle pricing issue is accounting properly for customers’ valuation of the bundle. As we have shown in this paper, if the bundle valuation is derived using the typical independence assumption, then the seller can forgo arbitrarily large profits. In this context, we have introduced and quantified the concept of price of independence, and shown how to estimate the parameters that capture the dependence relationship. We have also shown that simple price approximations are surprisingly effective in determining near-optimal profits, even when the dependence structure is accounted for.

The basis for our analysis has been the promising, yet little-used within marketing, framework of copula theory. A main benefit of this approach is that it can directly capture a dependence structure between the components for any marginal distribution. Besides the theoretical benefits that enabled us to derive optimal prices for the different bundling strategies, the copula-based framework has an even more practical benefit: it is an easier exercise for a seller to estimate (marginal) component valuations and a general dependence structure, than to estimate the joint component valuation. However, we should point out that, although we are optimistic about the proposed method and would argue to its strengths, copula theory and its application is not without its ardent critics. The use, misuse and abuse of copula theory has been feverishly discussed in both popular press [Whitehouse, 2005, Salmon, 2009] and academic research journals (e.g., the spirited debate in Extremes, vol.9 (1), 2006). In this area, our opinion and philosophy ascribes to Embrechts’ [2009] call to employ a balanced approach when dealing with sensitive problems. We also acknowledge that specification regarding which copula function to choose from in fitting data is a sensitive and important topic, but one that is not restricted to bundle pricing; see Durrleman et al. [2000] and Embrechts [2009] for other interesting examples.

The main research objective of this paper has been to derive optimal prices under the three predominant bundling strategies. To focus the discussion on the topic of dependence between the component valuations, we have kept many of the other important facets or
“dimensions” of bundling fixed. We hope that our work will motivate future research that explores even further the complex relationships between components and bundles.

Acknowledgment

The research project was partially funded by Professor Ødegaard’s NSERC Discovery Grant 372074-2009.

APPENDIX A: Estimating the Dependence Relationship

The purpose of this Appendix is to provide an overview of copula estimation and model fit. For more details and additional references, see the citations provided. Similar to simple or multivariate linear regression models where there exist various approaches for estimating the model parameters (e.g., ordinary least squares, maximum likelihood, method of moments, Bayesian inference, etc.), there are many different methods for estimation or statistical inference of copula models. Given the relative recent development of copula theory, naturally, there is no established consensus as to which method should be preferred or more efficient. Two factors that further complicate the statistical inference of copulas are: 1) the computational burden due to the multivariate nature of the models, and 2) that both the copula parameter \( \theta \) and the marginal distributions have to be estimated. To illustrate the difficulty of, for instance, Maximum Likelihood Estimation (MLE), consider the bivariate copula model

\[
F(x_1, x_2; \alpha_1, \alpha_2, \theta) = C(F_1(x_1; \alpha_1), F_2(x_2; \alpha_2); \theta),
\]

where \( F_1(\cdot; \alpha_1) \) and \( F_2(\cdot; \alpha_2) \) are the univariate marginal distributions with parameters \( \alpha_1 \) and \( \alpha_2 \), respectively, and \( C(\cdot; \theta) \) the copula function with parameter \( \theta \); the parameters are written in bold to highlight that they may be multidimensional. Suppose we have a sample of \( n \) i.i.d. pairs of observations \((x^k_1, x^k_2)\), \( k = 1, \ldots, n \), and wish to maximize the log-likelihood function

\[
L(\alpha_1, \alpha_2, \theta) = \sum_{k=1}^{n} \log[f_1(x^k_1; \alpha_1)f_2(x^k_2; \alpha_2)c(F_1(x^k_1; \alpha_1), F_2(x^k_2; \alpha_2); \theta)].
\]

Under usual regularity conditions, a solution to \( \partial L / \partial \alpha_1, \partial L / \partial \alpha_2, \partial L / \partial \theta = 0 \), provides consistent and asymptotic efficient maximum likelihood estimators [Joe, 1997, chapter 10]. It should be fairly clear that given the multitude of possible marginal distributions and
various copula functions that a single algorithm or approach will not work equally well. Therefore the specific approach for solving the MLE tends to be ad hoc. For examples pertaining to the Gaussian copula, see section 3 of Danaher and Smith [2011].

A simplification to the MLE method is to divide the estimation into a two-step method of Inference Functions for Margins (IFM) [Joe, 1997, chapter 10]. First, we estimate the parameter(s) $\alpha_i$ for the univariate marginal distributions by solving the MLE of $L_i(\alpha_i) = \sum_{k=1}^{n} \log f_i(x^k_i; \alpha_i), i = 1, 2$. In the second step we use the estimates $(\hat{\alpha}_1, \hat{\alpha}_2)$ to solve the MLE of $L_\theta(\theta) = \sum_{k=1}^{n} \log c(F_1(x^k_1; \hat{\alpha}_1), F_2(x^k_2; \hat{\alpha}_2); \theta)$. Again under regularity conditions, a solution to the system $(\partial L_1/\partial \alpha_1, \partial L_2/\partial \alpha_2, \partial L_\theta/\partial \theta) = 0$ provides consistent and asymptotic efficient maximum likelihood estimators. Note, however, that in general the two MLE procedures do not provide equivalent estimators [cf. Joe, 1997, pg. 300].

A simpler approach in estimating the copula parameter, at least for single-parameter copula functions, is based on the empirical rank correlations [Genest and Favre, 2007]. For instance, an asymptotically unbiased estimator of Kendall’s $\tau$ is given by $\hat{\tau} = 4P_n/[n(n - 1)] - 1$, where $P_n$ is the number of concordant pairs in the sample set $(x^k_1, x^k_2), k = 1, \ldots, n$. Let $I_{kl}$ be an indicator function such that $I_{kl} = 1$, if $x^k_1 < x^l_1, x^k_2 < x^l_2$ for $k \neq l$, and $I_{kl} = 0$ otherwise, then $P_n = \sum_{k}^{n} \sum_{l \neq k} I_{kl}$. The main benefit of this approach is that under the null hypothesis $H_0 : C = C_{\Pi}$ (i.e. that $X_1$ and $X_2$ are independent and $\tau = 0$), the estimate $\hat{\tau}$ follows a Normal distribution with mean zero and variance $2(2n + 5)/[9n(n - 1)]$ [Genest and Favre, 2007, pg. 351]. In other words, it is possible to compute a formal test statistic regarding the dependence between the two variables. Finally based on the estimated $\hat{\tau}$ it is straightforward to derive an estimate of the parameter $\theta$ for a given copula function. For example, given the basic principles discussed in Section 2, the FGM copula parameter is simply $\hat{\theta} = 9\hat{\tau}/2$, while an estimate of the Frank copula parameter is the solution to $\hat{\tau} = 1 + 4(D_1(\hat{\theta}) - 1)/\hat{\theta}$.

Given the previous discussion regarding parameter estimation it should be fairly unsurprising that formal goodness-of-fit tests are also a challenge. We again may note that both the marginal distribution model and the joint distribution copula model need to be assessed. For a review on the topic and power study comparison, see Genest et al. [2009]. Two semi-formal model comparison tests suggested by Joe [1997] include assessing the log-likelihoods or Akaike Information Criterion (AIC) values of different models, and comparing models based on predictive ability. Genest and Favre [2007] also provide a graphical method for assessment based on QQ-type plots. Finally, two formal goodness-of-fit tests are discussed in Genest et al. [2013] and Huang and Prokhorov [2014].
APPENDIX B: The effect of heavy-tailed component valuations

In Section 4.3 we observed that even under extreme marginal distributions with heavy or long-tails (e.g. Pareto and Weibull), the Price of Independence is still significant. This may at first seem a bit surprising since for heavy-tail distributions the difference in probability mass asymptotically diminishes between variables with independence and with (strong) dependence structure; see e.g. Theorem 2 in Ibragimov et al. [2015]. However, we should note that the PoI is with regard to the optimal prices, which for the cases analyzed in Section 4.3 are all found at relatively “low” prices and not in the right tail. Nonetheless, motivated by Ibragimov and Walden [2010] and Ibragimov et al. [2015] we were interested to further investigate the effect that heavy-tailed marginal distributions have on both demands and profits when prices are non-optimally set in the tail. Specifically, we were intrigued to see whether the demands/profits become “significantly” indistinguishable from the independence case as we price deeper and deeper into the tail of the distribution. To do this we extended the numerical analysis based on the Pareto(1,1) distribution and calculated the respective bundle demands and expected profits for three cases corresponding to negative, independence, and positive dependence captured by the FGM copula for non-optimal prices between 50 and 1000 in increments of 50 (that is, we went all the way from approximately the 95-th to the 99.8-th percentile of the distribution of $P(x_1 + x_2 > p_b|\theta), \theta \in \{-1, 0, 1\}$). The results are summarized below in Table 1. The last four columns represent the relative changes in demands and profits in comparing the case of $\theta = \pm 1$ with independence ($\theta = 0$), respectively. We used relative changes in order to be consistent with the PoI calculations in the paper.

Notice that while the demands ($Q_b$) are identical up to the third or fourth decimal digit (consistent with Theorem 2 in Ibragimov et al. [2015]), once we switch to profits ($\Pi$) the fourth decimal digit and beyond become important due to scaling up by price. For example, when $p = 100$, the demands when $\theta = -1$ or $\theta = 1$ are identical up to the third decimal with the demand when $\theta = 0$. When computing the profit by multiplying by 100, the profits are now different enough so that we have a non-optimal pricing profit gap of $(2.0919 - 2.0426)/2.0426 = 2.41\%$ or $(2.1412 - 2.20919)/2.0919 = 2.35\%$, respectively.

In conclusion, while one observes an asymptotically diminishing difference in demand, for a given price in the far right tail there is an observable difference in the expected profit between the dependent and independent cases.
Since both valuations are assumed IFR, then the profit is unimodal and admits the unique solution $p_i^* = g_i^{-1}(1)$. 

**Proof.** Proof of Lemma 2. Applying the FOC yields $Q_b + (p_b - m)\frac{dQ_b}{dp_b} = 0$. The result then follows by noting that $\frac{dQ_b}{dp_b}/Q_b = \frac{d}{dp_b} \ln Q_b$. 

**APPENDIX C: Proofs**

Proof of Proposition 1. The First Order Conditions (FOC) are $\frac{\partial \Pi}{\partial p_i} = 0$, which can be rewritten as $1 - F_i(p_i) - (p_i - m_i)f_i(p_i) = 0$, $i = 1, 2$. Dividing each side by $1 - F_i(p_i)$ yields the desired result. If $m_i = 0$ the functional equation reduces to $g_i(p_i) = 1$. Since both valuations are assumed IFR, then the profit is unimodal and admits the unique solution $p_i^* = g_i^{-1}(1)$. 

Proof of Proposition 3. We will prove only the first case, since the second follows immediately in an identical manner. Assume therefore that $0 < p_b < 1$. Substituting $C_1(x, y) = y[1 + \theta(1 - 2x)(1 - y)]$ in (6) yields 

$$Q_b = 1 - \frac{p_b^2(\theta + 1)}{2} - \frac{\theta p_b^2}{6} (4 - p_b).$$
We thus have
\[
\frac{d}{dp_b} \ln Q_b = \frac{2p_b[3 - \theta(2p_b^2 - 6p_b - 3)]}{p_b^2[3 - \theta(p_b^2 - 4p_b - 3)] - 6}.
\]
Applying the necessary condition from Lemma 2 leads to the desired result. Finally, the Abel-Ruffini Theorem guarantees the existence of an algebraic solution to the quartic equation.

Proof. Proof of Proposition 4. The single iteration Newton’s method computes the approximation
\[
p_b^{(1)} = p_b^{(0)} - \frac{z(p_b^{(0)})}{z'(p_b^{(0)})},
\]
where \(z(p_b)\) denotes the generic quartic equation from Proposition 3, and \(z'(p_b)\) is the first derivative of \(z\) with respect to \(p_b\). Substituting \(p_b^{(0)} = (m + \sqrt{m^2 + 6})/3\) and \(p_b^{(0)} = 2(m + 1)/3\), respectively, and simplifying yields the desired result. Notice that due to the explicit modeling of dependence, as well as the lack of any results concerning unimodality in the context of copula convolution, the first order conditions are necessary but not sufficient. However it is easy to verify numerically that for all \(0 \leq m \leq 1, -1 \leq \theta \leq 1\) the second derivative \(\partial^2 \Pi_{PB}/\partial p_b^2 < 0\), implying that the sufficient second order KKT conditions are also satisfied.

Proof. Proof of Proposition 6. Let \(H = F_1 + F_2\) denote the distribution function of the bundle valuation, i.e. \(Q_b = 1 - H(x)\). Frank et al. [1987] show that for any distributions \(F_1\) and \(F_2\), the following inequalities are sharp:
\[
H \leq H(z) \leq \overline{H}
\]
where
\[
\underline{H} = \sup_{x \in \mathbb{R}} \max \{F_1(x) + F_2(z - x) - 1, 0\}
\]
\[
\overline{H} = 1 + \inf_{x \in \mathbb{R}} \min \{F_1(x) + F_2(z - x) - 1, 0\}.
\]
We first manipulate the inequalities algebraically to obtain \(1 - \overline{H} \leq Q_b(z) \leq 1 - H\).
We then substitute $F_1(x) = F_2(x) = x$ yielding

$$-\min \{z - 1, 0\} \leq Q_b(z) \leq 1 - \max \{z - 1, 0\}$$

$$\Leftrightarrow -(z - m) \min \{z - 1, 0\} \leq \Pi_{PB} \leq (z - m)(1 - \max \{z - 1, 0\}).$$

We need to analyze two cases. First, suppose that $0 \leq z \leq 1$. Then, the inequality above resolves to

$$-(z - m) \min \{z - 1, 0\} \leq \Pi_{PB} \leq (z - m)(1 - \max \{z - 1, 0\}).$$

The second case, $1 < z \leq 2$, yields

$$0 \leq \Pi_{PB} \leq (z - m)(2 - z).$$

**Proof.** Proof of Proposition 7. Consider the profit gradient

$$\nabla \Pi(p_1, p_2, p_b|m_1, m_2, \theta) = (\partial \Pi/\partial p_1, \partial \Pi/\partial p_2, \partial \Pi/\partial p_b)^T$$

which can be explicitly rewritten as

$$\frac{\partial \Pi}{\partial p_1} = (p_1 - m_1) \frac{\partial Q_1}{\partial p_1} + Q_1 + (p_b - m_1 - m_2) \frac{\partial Q_b}{\partial p_1}$$

$$\frac{\partial \Pi}{\partial p_2} = (p_2 - m_2) \frac{\partial Q_2}{\partial p_2} + Q_2 + (p_b - m_1 - m_2) \frac{\partial Q_b}{\partial p_2}$$

$$\frac{\partial \Pi}{\partial p_b} = (p_1 - m_1) \frac{\partial Q_1}{\partial p_b} + (p_2 - m_2) \frac{\partial Q_2}{\partial p_b} + (p_b - m_1 - m_2) \frac{\partial Q_b}{\partial p_b} + Q_b.$$  

In general, the gradient ascent search optimizes the objective function iteratively, by moving from an initial point $p^{(0)} = (2/3, 2/3, (4 - \sqrt{2})/3)^T$ to successive solutions $p^{(k)}, k \geq 1$ in the direction of the gradient. Since we perform only a one-step ascent, we determine $p^{(1)}$ by solving $p^{(1)} = p^{(0)} + \nabla \Pi(p^{(0)}|m_1, m_2, \theta)$. This is a linear system in three unknowns, which can be solved analytically using determinants (e.g. Cramer’s rule). Substituting the analytical evaluations of the gradient into the linear system and solving for the prices yields our result.

Finally, the upper bound on the marginal costs is a consequence of the bundle price constraint $p_b \leq p_1 + p_2$, which guarantees the existence of a non-degenerate full bundling strategy. Substituting the approximate prices in the bundle price constraint and separating the marginal costs results in the condition $0 \leq m_1, m_2 \leq \frac{80 - 7\sqrt{2}}{120\sqrt{2}} \approx 0.4661$. Just like in the proof of Proposition 4, it is easy to verify numerically by substituting the approximate price vector $p$ into the Hessian matrix $H$ of the profit results in $pHp^T < 0$ for all $0 \leq m \leq 0.466, -1 \leq \theta \leq 1$, thus validating the second order KKT conditions.
For the proof of part b) of the proposition, consider the profit function $\Pi_{MB} = p_1Q_1 + p_bQ_b$, with the expressions for $Q_1$ and $Q_b$ as specified in Section 3.3. We then apply the same methodology as above.

References


