New results concerning probability distributions with increasing
generalized failure rates

Mihai Banciu                        Prakash Mirchandani
School of Management,              Katz Graduate School of Business,
Bucknell University                University of Pittsburgh
Lewisburg, PA, 17837              Pittsburgh, PA, 15260
mmb018@bucknell.edu                pmirchan@katz.pitt.edu

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Abstract

Lariviere and Porteus (2001) introduce the concept of generalized failure rate of a continuous random variable and demonstrate its importance. If the valuation distribution of a product has an increasing generalized failure rate (that is, the distribution is IGFR), then the associated revenue function is unimodal, and when the generalized failure rate is strictly increasing, the global maximum is uniquely specified. Assuming that the distribution is IGFR is thus useful and frequently-held in recent pricing, revenue and supply chain management literature. This note extends the IGFR concept in three ways. First, it investigates left and right truncations of valuation distributions; truncations arise when price floors or price ceilings are imposed. Second, it generalizes the IGFR definition to situations where the buyer’s payment is different from the seller’s receipt due to, say, taxes or subsidies. Third, we extend the IGFR notion to discrete distributions and contrast it with the continuous distribution case. The note also addresses two errors in the previous IGFR literature. Finally, we analyze all common (continuous and discrete) distributions for the prevalence of the IGFR property, and derive and tabulate their generalized failure rates, for future reference.

Keywords: probability distributions; hazard rate functions; revenue management; supply chain management
1 The Generalized Failure Rate

Given a nonnegative continuous random variable $X$, with support $L$ and $U$, $0 \leq L < U < \infty$, let $f_X(x)$ and $F_X(x)$ respectively denote its density and distribution functions. Denote by $\bar{F}_X(x) = 1 - F_X(x)$ the reliability function of $X$. Lariviere and Porteus (2001) define the generalized failure rate $g_X(x) = x h_X(x)$ where $h_X(x) = f_X(x) / \bar{F}_X(x)$ is the failure rate of $X$. The random variable $X$ is said to have the Increasing (or, Decreasing) Failure Rate (that is, IFR or DFR property) if $h_X(x)$ is weakly increasing (or, decreasing) in $x$ for all $x | F_X(x) < 1$. Likewise, $X$ has Increasing (or, Decreasing) Generalized Failure Rate (that is, IGFR or DGFR property) if $g_X(x)$ is weakly increasing (or, decreasing) in $x$ for all $x | F_X(x) < 1$. If the random variable $X$ is IFR (or, DFR, IGFR, DGFR), then we also say that the corresponding distribution function $F_X(x)$ is IFR (or, DFR, IGFR, DGFR respectively). Clearly, when $L \geq 0$, all IFR variables are also IGFR (Lariviere, 2006). Furthermore the IFR property is implied by the logconcavity of the associated density function (Bagnoli and Bergstrom 2005), therefore all logconcave random variables are also IGFR.

If $p$ denotes the price of a product, $\bar{F}_P(p)$ the corresponding demand, and $F_P(p)$ is IGFR, then the revenue function $\pi(p) = p \bar{F}_P(p)$ is unimodal (Lariviere 2006). If $F_P(p)$ has a strictly increasing generalized failure rate, then the revenue function admits a unique revenue-maximizing price. Consequently, the IGFR property is extremely useful in pricing, revenue management, and supply chain management research (e.g. Araman and Caldentey 2009, Chod and Rudi 2005, Kocabiyikoğlu and Popescu 2011, Paul 2005, Perakis and Roels 2007, Ziya et al. 2004).

Thus, determining whether a distribution satisfies the IGFR property is valuable, and researchers have studied the properties of IGFR distributions (Paul 2005, Lariviere 2006). This paper expands the scope of applicability of the IGFR property. One of the contributions of this paper is to establish the invariance of the IGFR property under both left and right truncations, thus permitting the use of IGFR results from the literature for situations where valuation functions need to be truncated (due to frequently imposed controls such as price floors and/or ceilings). We also model (the commonly occurring) situations where the buyer’s payout is different from the seller’s receipt. This phenomenon occurs, for example, with taxes and subsidies; we investigate such transformations in the context of IGFR distributions.

Previous IGFR work has considered continuous distributions only. We extend the concept of
generalized failure rate to discrete distributions, which is important for several reasons. Discrete demand is a natural phenomenon in situations where the product is indivisible or the buyers are atomic. In such cases, assuming a continuous distribution can lead to erroneous results. For example, Bagnoli et al. (1989) show that in a durable goods monopolistic setting, important results in widely-cited papers (Bulow 1982, Stokey 1979) for the continuous case do not hold when the demand distribution is discrete. Discrete demand situations also arise when the product valuations are discrete. For example, in the case of certain auctions, (e.g. oral auctions (Rothkopf and Harstad 1994)), the bid increments are discrete and so the valuations are also implicitly discrete. Therefore, it is important to investigate the IGFR property in a discrete setting. Interestingly, unlike for the continuous case, we find that the IGFR property for a discrete random variable does not imply that the associated revenue function is unimodal. However, we develop two sufficient conditions, both of which imply unimodality of the revenue function with discrete valuations. (One of these conditions slightly modifies the generalized failure rate function.) We also provide an optimality condition that holds for revenue maximization in the discrete distribution case.

We also examine Paul’s (2005) results considering the closure properties of IGFR distributions and correct two errors in his analysis—relating to closure under addition and to identification of DGFR distributions. We summarize these corrections in the Appendix and also provide sufficient conditions for the sum of two independent IGFR variables to be IGFR. As has been noted previously in the literature (Paul, 2005), this closure property is useful in situations where we aggregate independent demands, as in the case of a distributor who serves multiple retailers. Finally, we investigate the prevalence of the IGFR property for a large number of common (continuous and discrete) probability distributions, and develop and catalog their generalized failure rates.

2 New Results about the IGFR Property

The IGFR property is valuable for research in pricing and supply chain optimization. In this section, we present several new results about the IGFR class of distributions. These results broaden the applicability of the IGFR assumption to more general situations.
2.1 IGFR preservation under truncation

As previously observed in the literature, and as the comprehensive list in Section 3 shows, a large number of distributions are IGFR. However, we may need to model a truncated distribution where the support is restricted to be a subset of the support for the original distribution, and the relative probability densities of any two points in this subset for the truncated and the original distributions are equal. For example, the standard Normal distribution is IGFR over the nonnegative domain; however, the seller may want to place a lower and/or an upper bound on the charged price and hence she needs to work with a truncated standard Normal distribution.

Price ceilings and price floors are often used to regulate trade in important ways. For example, the European Union has price ceilings on mobile roaming charges for phone calls\(^1\). China has a price ceiling on gasoline\(^2\) (and so did the US in 1971). Rent control acts in many cities impose a ceiling on maximum increases in the rent. Price ceilings correspond to a right truncation of the distribution function and prevent price gouging, curb inflation and monopolistic power, and may provide economic help to lower income citizenry.

Similarly, we note that a left truncation (that is, a price floor) might result from a government policy to induce behavior change\(^3\) or prevent predatory pricing\(^4\), by an industry to promote fair trade\(^5\), or by a company to boost its bottom line\(^6\).

Assume that \(f_X(x)\) is differentiable, and denote the first and second derivatives of \(f_X(x)\) by \(f'_X(x)\) and \(f''_X(x)\), respectively. The following proposition establishes that the IGFR property is closed under both right and left truncations.

**Proposition 1.** Let \(X\) be a continuous IGFR random variable with support on the interval \([L, U] \subseteq \mathbb{R}\). Then, for any \(a\) and \(b\) such that \(L \leq a < b \leq U\), the truncation of \(X\) defined on the sub-interval \([a, b]\) is also IGFR.


\(^5\)http://www.fairtradeusa.org/press-room/press_release

Proof. The truncated cumulative density function is $F_{tr}(x) = [F_X(x) - F_X(a)]/[F_X(b) - F_X(a)]$ and the truncated density function is $f_{tr}(x) = f_X(x)/[F_X(b) - F_X(a)]$. Therefore, the generalized failure rate of the truncated $X$ variable is $g_{tr}(x) = xf_X(x)/[F_X(b) - F_X(a)]$, and

$$g'_{tr}(x) = \frac{(F_X(b) - F_X(x)) (xf'_X(x) + f_X(x)) + x(f_X(x))^2}{(F_X(b) - F_X(x))^2}.$$  

If $xf'_X(x) + f_X(x) \geq 0$, then clearly $g'_{tr}(x) \geq 0$. On the other hand, if $xf'_X(x) + f_X(x) < 0$, then

$$g'_{tr}(x) \geq \frac{(1 - F_X(x)) (xf'_X(x) + f_X(x)) + x(f_X(x))^2}{(1 - F_X(x))^2} = g'_X(x) \geq 0,$$

where the last inequality follows from the hypothesis.

**Proposition 2.** Let $X$ be a continuous IGFR random variable, and for any $a$ such that $L \leq a < U$, let $SI$ be the sub-interval $[a,U]$. Then, the generalized failure rate of $X$ over the sub-interval $SI$ and that of the left truncation of $X$ defined on $SI$ are the same.

**Proof.** Follows from substituting $F(b) = F(U) = 1$ in the proof above.

**Remark 3.** Given the revenue maximizing price, $x^*$ for an IGFR valuation function over the support $[L,U] \subseteq \mathbb{R}^+$, an implication of Proposition 2 and Remark 1 of Lariviere (2006) is that the revenue maximizing price over the support $[a,U]$, $a \geq L$, is $\max(x^*,a)$.

### 2.2 An Extension of Lariviere and Porteus’s Generalized Failure Rate Function

Paul (2005) shows that if a generalized failure rate function $g_X(x)$ is increasing then so is the generalized failure rate function $g_Y(y)$ where $y = ax^k$ for any $a,k > 0$. This transformation is intuitive, and is applicable, for example, in situations where the seller raises or drops the selling price by a non-zero, positive multiplier, or accounts for price inflation or deflation. However, this transformation assumes that the transaction price for both the buyer and the seller is the same.

We consider a different transformation of $X$ which results in an extension of Lariviere and Porteus’ (2001) generalized failure rate function. To motivate this definition, note that the unit purchase price may be different from the unit revenue generated by the seller. This difference
may arise due to sales taxes, subsidies, tips at restaurants, medical copays, commissions (such as those charged by Ebay), shipping charges, Federal rebates (such as those for the Cash for Clunkers program, and for those currently available on the Chevy Volt and energy efficient appliances). Thus, the buyer may pay an amount $x$ for a product but the seller may get $u(x)^8$. Clearly, fundamental economic principles dictate that $u(x)$ be non-negative and increasing in $x$.

In this situation, the revenue function is $\pi(x) = u(x)F_X(x)$ and the first order optimality condition is $[u(x)/u'(x)]h_X(x) = 1$. Thus, this alternate definition of the generalized failure rate is $g_X^E(x) = [u(x)/u'(x)]h_X(x)$. Observe that if $u(x) = x$, then the two generalized failure rate definitions are identical. Note that concavity of $\pi(x) = xF(x)$ is equivalent to $g_X^E(x)$ increasing in $x$ (see Lariviere, 2006, for a similar condition for concavity of $\pi(x)$). In Remark 4 below, we discuss conditions needed for the optimal solution $x^*$ to be an interior solution in the domain $[L, U]$.

(Per convention, we omit “weakly” when referring to $h_X(x)$ or $g_X(x)$ in the following discussion.)

For any function $t(x)$, let $\epsilon(t,x) = xt'/t(x)$ denote the elasticity of $t(x)$ with respect to $x$, and $\epsilon'(t,x) = \frac{d}{dx}\epsilon(t,x)$. If $\epsilon'(t,x) \leq 0$, we say that $t$ has nonincreasing elasticity.

**Proposition 4.** Let $u(x)$ be a logconcave, twice differentiable function of $x$. Then $g_X^E(x)$ is an increasing function of $x$ if either of the following conditions hold: a) $X$ is IFR; b) $X$ is IGFR and $\epsilon'(u,x) \leq 0$.

**Proof.** a) If $[(u(x)/u'(x)] \geq 0$ then $g_X^E(x)$ would be increasing whenever $X$ is IFR. However, $[(u(x)/u'(x)' \geq 0$ whenever $u(x)$ is logconcave. Hence the result follows.

b) If $[(u(x)/(xu'(x))]' \geq 0$ then $g_X^E(x)$ would be increasing whenever $X$ is IGFR. However, $[(u(x)/(xu'(x))]' \geq 0$ if and only if $u(x)$ has nonincreasing elasticity, that is, $\epsilon'(u,x) \leq 0$.

**Remark 5.** Compared to the situation when $X$ is IFR, Proposition 4 requires the additional condition $\epsilon'(u,x) \leq 0$ when $X$ is IGFR. Let us check this additional condition for three common transformations: linear, power, and logarithmic. First note that the linear transformation $u(x) = ax + b$ is increasing for $a > 0$, and nonnegative for $x \geq b/a$. It is also logconcave and twice differentiable. The additional sufficiency condition resolves to $b \leq 0$. The power function $u(x) = x^a$ is logconcave and twice differentiable for $a > 0$. Therefore, $\epsilon'(u,x) \leq 0$ if $a \leq 1$.


\footnote{We can set up the revenue function equivalently as $yF_Y(y) = yF_X(u^{-1}(y))$ where $y$ is the seller’s unit revenue; it is easy to show that $g_Y^G(y) = g_X^G(x)$.}
ax^k$ for $a, k > 0$ is also nonnegative, increasing in $x$, logconcave and twice-differentiable. Since $\epsilon(ax^k, x) = k$, the additional sufficiency condition always holds. The logarithmic transformation $u(x) = a \log(x)$, $a > 0$ and $x \geq 1$ is nonnegative, increasing in $x$, logconcave and twice-differentiable. (Note that Proposition 1 implies that if $X$ is IGFR under the nonnegative domain then it is also IGFR for $x \geq 1$.) In this case, the additional sufficiency condition is equivalent to the requirement that $1/\log(x)$ is nonincreasing, which always holds.

**Remark 6.** We can write equivalently $g_E^X(x) = [u(x)/u'(x)] h_x(x) = |\epsilon(D, x)|/\epsilon(u, x)$ where $\epsilon(D, x)$ is the price elasticity of demand (which is negative for normal goods); note that $\epsilon(u, x) \geq 0$ by our assumption on $u(x)$). The ratio of the two elasticities (adjusting for the negative sign) equals one at the optimal $x$. As an example (see Figure 1 below), consider $\bar{F}(x) = 1 - x$ and $u(x) = x + b, 0 < b < 1$. This corresponds to the dealer holdback situation in US automotive retailing where the buyer pays an amount $x$ and the dealer gets an amount $x + b$ due to the manufacturer’s payment to the dealer following the vehicle sale. The optimal price satisfying the condition $g_X(x^*) = |\epsilon(D, x^*)| = 1$ is $1/2$, and the optimal price satisfying $g_E^X(x^*) = 1$ is $(1 - b)/2$. Thus, the optimal price charged to the buyer with the transformation is smaller than in the original scenario. Observe that in this case, $\epsilon(u, x) = x/(x + b) < 1$ for all $x$. Since at the optimal point $x^*$, $|\epsilon(D, x^*)| = \epsilon(u, x^*)$, we get $|\epsilon(D, x^*)| < 1$ with the transformation. Moreover, the optimal price is lower than $1/2$, and for these illustrative parameters, the manufacturer-subsidy-to-the-seller $b$ is shared equally by the buyer and the seller\(^9\).

**Remark 7.** Note that we can also write $g_E^X(x) = |\epsilon(D, x)|/\epsilon(u, x)$ as $|\epsilon(D, u)|$. We can think of $|\epsilon(D, u)|$ as the *transactional* elasticity of demand, that is, the ratio of the relative change in end-customer demand and the relative change in the seller’s receipt. The seller’s receipt could be different from the buyer’s payment due to transactional distortions. Our optimality condition becomes $|\epsilon(D, u)| = 1$. If $|\epsilon(D, u(L))| > 1$, then the seller sets the optimal price at $L$ (because $u(x)$ is increasing in $x$). Likewise, if $|\epsilon(D, u(U))| \leq 1$, then the seller will set the optimal price at $U$. Thus, for interesting cases, we need to assume that $\lim_{x \downarrow L} g_E^X(x) \leq 1$ and $\lim_{x \uparrow U} g_E^X(x) > 1$.

\(^9\)One of the co-authors faced the opposite situation in 2009 when buying energy-efficient replacement windows, for which a tax credit to the buyer was in effect. The home improvement store was not willing to give their usual discount.
2.3 The Generalized Failure Rate for Discrete Distributions

To the best of our knowledge, the generalized failure rate has been defined and studied only for continuous distributions. While continuous distributions may also give us differentiability and thus enhanced tractability, as previous research illustrates (see the introduction section), results derived assuming continuous demand distributions need not apply for discrete distributions.

For completeness, we extend Lariviere and Porteus’ (2001) concept to a discrete, nonnegative random variable, $X = \{x_i\}_{i \in I}$ where $I = \{1, 2, \ldots, I_{\text{max}}\}$ and $0 \leq x_i \leq x_{i+1}$ for $i \in I \setminus I_{\text{max}}$. ($I_{\text{max}}$ can go to infinity.) Let $P_X(i)$ denote the probability for $X = x_i$. $X$ is logconcave, or equivalently that $P_X(i)$ is logconcave, if $P_X(i) \geq P_X(i - 1)P_X(i + 1)$ (see Hoggar 1973, Keilson and Gerber 1971). Hoggar (1973) showed that the sum of two independent discrete, logconcave random variables is logconcave. The failure rate for a discrete random variable (Barlow et al. 1963) is defined as $h_X(i) = P_X(i)/\sum_{k \geq i} P_X(k)$. Keilson and Gerber (1971) show that if $X$ is logconcave, then it is IFR. When a constant separates successive elements of the support, we say that $X$ is defined on a (one-dimensional) connected lattice.

In this discrete setting, we define $g_X(i) = x_iP_X(i)/\sum_{k \geq i} P_X(k)$. Clearly, if $X$ is IFR then it is also IGFR. Therefore, since the Bernoulli, Binomial, Poisson, and discrete Uniform distributions are logconcave, they are also IGFR. Note that all these distributions are defined on a lattice. Besides
the discrete distributions mentioned above that are IGFR (see also Section 3), there is an easy sufficiency condition that we can use to check for the IGFR characteristic for a general discrete distribution. This condition parallels Part b) of Corollary 1 in Lariviere (2006).

**Proposition 8.** Let \( X \) be a discrete random variable satisfying \( x_i P_X(i) \leq x_{i+1} P_X(i+1) \) for \( i \in I \setminus \{I_{\text{max}}\} \). Then, \( X \) is IGFR.

**Proof.**

\[
g_X(i) = \frac{x_i P_X(i)}{\sum_{k:k \geq i} P_X(k)} \leq \frac{x_{i+1} P_X(i+1)}{\sum_{k:k \geq i+1} P_X(k)} = g_X(i+1).
\]

\(\square\)

The antecedent in Proposition 8 is not necessary for \( X \) to have the IGFR property. For example, if the support of \( X \) is \( \{1, 2, 3, 4\} \) and corresponding probabilities are \( \{1/3, 1/8, 5/24, 1/3\} \), then \( X \) is IGFR but \( x_1 P_X(1) > x_2 P_X(2) \).

**Comparison with the Continuous Case**

Not surprisingly, as in the case of continuous random variables, discrete random variables that are not IFR may or may not be IGFR. For example, let the support of \( X \) be \( \{1, 2, 3, 4\} \). When the corresponding probability function is \( \{1/3, 1/8, 5/24, 1/3\} \), \( X \) is not IFR but it is IGFR, and when the probability function is \( \{1/3, 1/10, 7/30, 1/3\} \), \( X \) is neither IFR nor IGFR.

However, unlike the continuous case, an IGFR \( X \) does not guarantee that the corresponding revenue function \( x_i \sum_{k:k \geq i} P_X(k) \) is unimodal. For example, if \( X \) is supported on \( \{1, 2, 3, 4\} \), with probabilities \( \{0.1, 0.31, 0.14, 0.45\} \), then \( X \) is IGFR but the corresponding revenue function is bimodal.

Moreover, although the generalized failure rate equals one at the optimal revenue-maximizing point for continuous distributions (see Lariviere, 2006 for details), this property does not hold for discrete distributions. In fact, the optimal solution need not even be in the proximity of the points for which the respective generalized failure rates adjoin one. Consider the case when the support of \( X \) is \( \{1, 2, 3, 4, 5, 6, 7\} \) and the distribution is \( \{0.01, 0.24, 0.19, 0.115, 0.08, 0.06, 0.305\} \). The optimal price is \( X = 3 \), while the generalized failure rate crosses one between \( X = 6 \) and \( X = 7^{10} \).

\(^{10}\)We have used an example with a wide support just to emphasize the observation we are making.
Finally, a discrete random variable can never be DFR or DGFR. $X$ cannot be DFR since $h_X(x_1) = P_X(1) < 1$ and $h_X(x_{I_{\text{max}}}) = 1$. Likewise, since $x_1 < x_{I_{\text{max}}}$ and $P_X(1) < 1$, $g_X(1) = x_1P_X(1) < x_{I_{\text{max}}} = g_X(I_{\text{max}})$, $X$ cannot be DGFR either.

We now derive two different sufficiency conditions, each of which implies a unimodal revenue function. The first of these conditions requires a slight modification of the definition of the IGFR function; furthermore, when this condition holds, the optimality check is easy and corresponds to the optimality check mentioned above for continuous distributions. Like Keilson and Gerber (1971), we assume that $X$ is defined on a connected lattice for the rest of this subsection. Let $\delta > 0$ denote the (finite) increment between successive values in the support of $X$, and define $\gamma_X(i) = x_{i+1}h_X(i)$. When $I_{\text{max}} < \infty$ we set $\gamma_X(I_{\text{max}}) = \infty$. (Observe that $\gamma_X(i)$ is a slight modification of the generalized failure rate function.)

**Proposition 9.** Let $X$ be a discrete random variable defined on a connected lattice such that $\gamma_X(i)$ is increasing. Then the revenue function corresponding to $X$ is unimodal.

**Proof.** Assume that the revenue function is not unimodal. Then (i) $I_{\text{max}} \geq 3$, and (ii) there exists an index $m, m \in I \setminus \{1, I_{\text{max}}\}$, such that $x_{m-1}\sum_{j \geq m-1} P_X(j) > x_m\sum_{j \geq m} P_X(j)$ and $x_m\sum_{j \geq m} P_X(j) < x_{m+1}\sum_{j \geq m+1} P_X(j)$, with at most one of the two strict inequalities being an equality (for which case, the proof is similar). The first inequality implies

$$x_{m-1}\sum_{j \geq m-1} P_X(j) > x_m\sum_{j \geq m} P_X(j) \iff x_{m-1}\sum_{j \geq m-1} P_X(j) - (x_{m-1} + \delta)\sum_{j \geq m} P_X(j) > 0$$

$$\iff \gamma_X(m-1) > \delta.$$

On the other hand, the second inequality implies

$$x_m\sum_{j \geq m} P_X(j) < x_{m+1}\sum_{j \geq m+1} P_X(j) \iff x_m\sum_{j \geq m} P_X(j) - (x_m + \delta)\sum_{j \geq m+1} P_X(j) < 0$$

$$\iff \gamma_X(m) < \delta.$$

Since by hypothesis, $\gamma_X(i)$ is increasing, a contradiction occurs.

**Remark 10.** Since $\gamma_X(i) = g_X(i) + \delta h_X(i)$, Proposition 9 asserts the unimodality of revenue if “$g_X(i) + \delta h_X(i)$ is increasing” rather than the condition “$g_X(x)$ is increasing” as for the continuous
Proposition 11. Let \( X \) be a discrete random variable defined on a connected lattice such that \( \gamma_X(i) \) is increasing. Then the revenue is maximized at index \( i^* = \inf \{ i : \gamma_X(i) \geq \delta \} \).

Proof. Since \( \gamma_X(i) \) is increasing in \( i \), the revenue function is unimodal (Proposition 9). Now consider two adjacent indices \( i \) and \( i + 1, 1 \leq i < I_{\text{max}} \). Clearly, an optimal \( i \) is the smallest index for which the forward difference in the revenues, \( x_i \sum_{j \geq i} P_X(j) - x_{i+1} \sum_{j \geq i+1} P_X(j) \geq 0 \). Simplifying the left hand side leads to the desired condition. To show the existence of an optimal index \( i \), it is sufficient to note that \( \gamma_X(I_{\text{max}}) = \infty \).

A large number of well-known discrete distributions, including the Binomial, Negative Binomial, Geometric, and the Poisson distributions, satisfy \( \delta = 1 \); see also Johnson et al. (2005).

We now establish that the antecedent condition in Proposition 8 is sufficient for proving unimodality of the associated revenue function.

Proposition 12. Let \( X \) be a discrete random variable defined on a connected lattice such that \( x_i P_X(i) \leq x_{i+1} P_X(i + 1) \) for \( i \in I \setminus \{ I_{\text{max}} \} \). Then, the revenue \( x_i \sum_{j \geq i} P_X(j) \) is unimodal.

Proof. To show sufficiency of the given conditions, suppose the revenue function is not unimodal. Then (i) \( I_{\text{max}} \geq 3 \), and (ii) there exists an index \( m \) such that

\[
x_{m-1} \sum_{j \geq m-1} P_X(j) > x_m \sum_{j \geq m} P_X(j) < x_{m+1} \sum_{j \geq m+1} P_X(j)
\]

with at most one of the two strict inequalities being an equality (for which case, the proof is similar). Note

\[
x_{m-1} \sum_{j \geq m-1} P_X(j) > x_m \sum_{j \geq m} P_X(j) \iff x_{m-1} P_X(m-1) > (x_m - x_{m-1}) \sum_{j \geq m} P_X(j)
\]

and

\[
x_m \sum_{j \geq m} P_X(j) < x_{m+1} \sum_{j \geq m+1} P_X(j) \iff x_m P_X(m) < (x_{m+1} - x_m) \sum_{j \geq m+1} P_X(j)
\]

Moreover, \( x_{m-1} P_X(m-1) \leq x_m P_X(m) \) and \( \sum_{j \geq m} P_X(j) \geq \sum_{j \geq m+1} P_X(j) \); thus \( 2x_m < x_{m-1} + x_{m+1} \). A contradiction ensues since by hypothesis the support of \( X \) is a lattice.
Remark 13. The two sufficiency conditions in Propositions 9 and 12, respectively, do not imply one
another. To see this, consider two examples for both of which the support of $X$ is \{1, 2, 3, 4\}. If the
corresponding probabilities are \{0.1, 0.5, 0.3, 0.1\}, then $\gamma_X(i)$ is increasing, but $x_2P_X(2) > x_3P_X(3)$.
Conversely, if the distribution is \{0.1, 0.055, 0.2, 0.645\}, then $x_iP_X(i) \leq x_{i+1}P_X(i+1)$ for $1 \leq i \leq 3$,
but $\gamma_X(1) = 0.2$ and $\gamma_X(2) = 0.183$.

2.4 Preservation under convolution

Using a counter example, Paul (2005) demonstrates that IGFR distributions are not closed under
addition. As we show in the Appendix, this example is not correct; we also provide a different
example that does support Paul’s claim. Paul concludes his note by stating “the fact that IGFR
distributions are not closed under convolution may be something of a limitation in supply chain
models, because it implies that pooling independent demands may destroy the IGFR property at
the level of aggregate demand.” We would like to point out that the situation is not as bleak as this
statement might indicate. Since logconcavity is closed under addition and logconcave distributions
are IFR, they are also IGFR. Thus, for example, the convolutions of standard Uniform, the standard
Normal, the Exponential, the Logistic, the Laplace, the Chi-squared with at least two degrees of
freedom, yield logconcave distributions. (See Bagnoli and Bergstrom (2005) for a comprehensive list
of logconcave distributions.) Thus, the original pair of distributions is IGFR and their convolution
is also IGFR. In addition, the convolution of two distributions that are IGFR but not IFR might
still be IGFR if the distributions are i.i.d. and stable. By definition, a random variable is stable
Johnson et al. (1994) if a linear combination of two independent copies of the variable has the same
distribution as the original (with differences at most in the scale and location parameters). For
example, the Cauchy and Lévy distributions are neither IFR nor DFR, but both are IGFR. Since
both distributions are stable, the IGFR property will be preserved if adding two Cauchy or two
Lévy random variables.

We now consider another situation, where the IGFR property is preserved under addition. If
$X$ and $Y$ are two IFR random variables, then Theorem 1 of Lariviere (2006) shows that $e^X$ and $e^Y$
are IGFR. Proposition 14 describes a sufficient condition for $Z = e^X + e^Y$ to be IGFR.

Definition. A function $K$ defined on $A \times B \subseteq \mathbb{R}^2$ is called TP$_2$ (totally positive of order 2) if for
all $x_1 < x_2 \in A$ and $y_1 < y_2 \in B$, $K(x_1, y_1)K(x_2, y_2) - K(x_1, y_2)K(x_2, y_1) \geq 0$. 

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Proposition 14. Let $X$ and $Y$ be two continuous independent logconcave random variables (and therefore IGFR), with probability density functions $f_X(\xi)$ and $f_Y(\xi)$ respectively. If both $f_X(\log \xi)/\xi$ and $f_Y(\log \xi)/\xi$ are logconcave, then, $Z = e^X + e^Y$ is IFR (and therefore IGFR).

Proof. The probability density function of $Z$ is given by

$$f_Z(z) = \int_{-\infty}^{\infty} \xi^{-1}(z - \xi)^{-1} f_X(\ln \xi) f_Y(\ln(z - \xi)) d\xi$$

From the definition of the $TP_2$ property, it follows that a function $K(x,y) = k(y - x)$ is $TP_2$ if and only if it is logconcave (Karlin 1968). The Basic Composition Theorem (Karlin 1968) establishes that if $K(x,\xi)$ is $TP_m$, $L(\xi,y)$ is $TP_n$, and $\sigma(\xi)$ is a $\sigma$-finite measure, then the convolution $M(x,y)$ is $TP_{\min\{m,n\}}$, where $M(x,y) = \int K(x,\xi) L(\xi,y) d\sigma(\xi)$.

Let $\hat{f}_i(x) = x^{-1} f_i(\ln x)$, $i = \{X,Y\}$. Then, $f_Z(z) = \int_{-\infty}^{\infty} \hat{f}_X(\xi) \hat{f}_Y(z - \xi) d\xi$. Separately, let $K(x,\xi) = \hat{f}_X(x - \xi)$ and $L(\xi,y) = \hat{f}_Y(\xi - y)$. Since these functions are assumed logconcave, then $K$ and $L$ are $TP_2$, and by the Basic Composition Theorem

$$M(x,y) = \int_{-\infty}^{\infty} \hat{f}_X(x - \xi) \hat{f}_Y(\xi - y) d\xi$$

is also $TP_2$.

However, note that by substituting $u = \xi - y$,

$$M(x,y) = \int_{-\infty}^{\infty} \hat{f}_X(x - y - u) \hat{f}_Y(u) du = f_Z(x - y)$$

Since $M$ is $TP_2$, then $f_Z(z)$ is logconcave. Hence, $Z$ is IFR. \qed

As an example, let $X$ be a random variable from the Gumbel family, characterized by $f(x) = \exp\{-e^x + x\}$. Since $f(x)$ is logconcave, then $X$ is IFR, so $Y = e^X$ is IGFR. But the density of $Y$ is $f(y) = y^{-1} \exp\{-e^{\log y} + \log y\} = e^{-y}$. This is a standard Exponential distribution ($\lambda = 1$), which is IGFR. The sum of two standard Exponential distributions is a Gamma distribution which is IGFR.

Figure 2 illustrates that the convolution of two standard Log-Normal variables is IGFR, and so there are other conditions that guarantee the closure of the IGFR property under addition. Finally, we note that even if not all IGFR distributions maintain closure under addition, the Central Limit
Theorem suggests that as the number of terms involved in the addition increase (and approach infinity in the limit), the overall convolution converges in distribution to a Normal form. Since the Normal distribution is IFR, in the limit, the sum will then also be IFR and therefore IGFR. Hence, from a modeling perspective, while there may some restrictions on distributional assumptions while aggregating at a micro-level (say, centralizing the profits in a supply chain with two suppliers), these restrictions become less and less restrictive as the aggregation moves to a macro-level (say, aggregating across an entire industry or economy).

3 Catalog of IGFR Distributions

We conclude this note by including, for completeness, a list of other known IFGR distributions. Tables 3 and 3.2 respectively tabulate the most widely used continuous and discrete IGFR distributions, together with the analytical forms (if expressible in closed form). An asterisk indicates that $F_X(x)$ or $g_X(x)$ does not have a closed form representation. These tables should prove useful for future econometrical modeling and pricing research.

All distributions presented below are instantiated by a location parameter $A$ and a scale parameter $B$. If the distribution is IFR, it is presented in standard form (i.e., $A = 0$ and $B = 1$), since IFR distributions maintain closure under scaling and shifting (Bagnoli and Bergstrom 2005).
Some IGFR distributions do not maintain the property for all combinations of location and scale parameters; in these cases we use the terminology $z(x) = (x - A)/B$ for brevity in Table 3. The notes in Section 3.1 provide additional information about the parameters. Where needed, the original domain has been truncated to reflect realistic support for the relevant random variable (e.g. distributions with support on the entire real axis have been truncated to the $[0, \infty)$ interval), and the densities have been adjusted accordingly to reflect the truncation. For distributions with support on the real axis (e.g. Normal, Logistic, etc.), Proposition 1 establishes that if the original density has the IGFR property, so does its truncation. In addition, Proposition 2 establishes the identity between the generalized failure rate of the left truncation and that of the original variable. For definitions and applications of each distribution, see Johnson et al. (1994, 1995) and Johnson et al. (2005).

3.1 Remarks on specific continuous distributions

The Uniform distribution: The distribution is defined on the interval $[a, b]$ and it is logconcave, and therefore IFR.

The Exponential distribution: The distribution is both logconcave and logconvex, therefore both IFR and DFR. The failure rate function is constant.

The Normal distribution: The distribution is logconcave, therefore IFR. In Table 3, $\Phi(x)$ denotes the standard Normal distribution.

The Logistic distribution: The distribution is logconcave, therefore IFR. Introducing a shape parameter $C > 0$ generalizes this distribution. The distribution function of this generalized logistic distribution (also known as a Type I Generalized Logistic distribution or "skew-logistic" distribution), with location parameter $A$ and scale parameter $B$ is

$$F(x) = \frac{1}{(1 + \exp\{(A - x)/B\})^C}.$$ 

The skew-logistic distribution is also IFR since

$$(\log(F'(x)))'' = -(C + 1)B^{-2}\exp\{(x + A)/B\}\{\exp\{A/B\} + \exp\{x/B\}\}^{-2} \leq 0$$
<table>
<thead>
<tr>
<th>Distribution</th>
<th>$f_X(x)$</th>
<th>$F_X(x)$</th>
<th>$g_X(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uniform $[a, b]$</td>
<td>$1/(b-a)$</td>
<td>$x/(b-a)$</td>
<td>$x/(b-x)$</td>
</tr>
<tr>
<td>Exponential $[0, \infty]$</td>
<td>$\lambda e^{-\lambda x}$</td>
<td>$1 - e^{-\lambda x}$</td>
<td>$\lambda x$</td>
</tr>
<tr>
<td>Normal $[0, \infty]$</td>
<td>$\frac{2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$</td>
<td>$2\Phi(x) - 1$</td>
<td>$\frac{x}{\sqrt{2\pi}[1-\Phi(x)]}e^{-\frac{x^2}{2}}$</td>
</tr>
<tr>
<td>Logistic $[0, \infty]$</td>
<td>$\frac{2}{1+e^{-x}}$</td>
<td>$\frac{2}{1+e^{-x}} - 1$</td>
<td>$\frac{1}{1+e^{-x}}$</td>
</tr>
<tr>
<td>Laplace $[0, \infty]$</td>
<td>$\frac{e^{-x/2}}{2}$</td>
<td>$1 - e^{-x/2}$</td>
<td>$x$</td>
</tr>
<tr>
<td>Weibull $[0, \infty]$</td>
<td>$k x^{k-1} \exp{-x^k}$</td>
<td>$1 - \exp{-x^k}$</td>
<td>$k x^k$</td>
</tr>
<tr>
<td>Gumbel Max $[0, \infty]$</td>
<td>$e^{-x} \exp(-e^{-x})$</td>
<td>$\exp(-e^{-x}) - e^{-1}$</td>
<td>$\frac{x e^{-x}}{\exp(e^{-x}) - 1}$</td>
</tr>
<tr>
<td>Gumbel Min $[0, \infty]$</td>
<td>$e^x \exp{1 - e^x}$</td>
<td>$1 - \exp{1 - e^x}$</td>
<td>$x e^x$</td>
</tr>
<tr>
<td>Gamma</td>
<td>$\frac{x^{k-1} e^{-x}}{\Gamma(k)}$</td>
<td>$*$</td>
<td>$*$</td>
</tr>
<tr>
<td>Beta $[0, 1]$</td>
<td>$x^{a-1} (1-x)^{b-1}$</td>
<td>$*$</td>
<td>$*$</td>
</tr>
<tr>
<td>Power Function $[0, 1]$</td>
<td>$k x^{k-1}$</td>
<td>$x^k$</td>
<td>$k x^k$</td>
</tr>
<tr>
<td>$\chi^2 [0, \infty]$</td>
<td>$\frac{x^{z-1} e^{-\frac{x}{2}}}{2^z \Gamma(z/2)}$</td>
<td>$*$</td>
<td>$*$</td>
</tr>
<tr>
<td>Chi $[0, \infty]$</td>
<td>$\frac{x^{k-1} e^{-\frac{x}{2}}}{2^k \Gamma(k)}$</td>
<td>$*$</td>
<td>$*$</td>
</tr>
<tr>
<td>Pareto I $[1, \infty]$</td>
<td>$k x^{-k}$</td>
<td>$1 - x^{-k}$</td>
<td>$k$</td>
</tr>
<tr>
<td>Pareto II $[A, \infty]$</td>
<td>$(k/B)(1 + z(x))^{-k-1}$</td>
<td>$1 - (1 + z(x))^{-k}$</td>
<td>$k x/(x + B - A)$</td>
</tr>
<tr>
<td>Burr $[A, \infty]$</td>
<td>$c k \frac{z(x)^{k-1}}{B[1+z(x)]^{k+c}}$</td>
<td>$1 - \left[1 + z(x)^k\right]^{-c}$</td>
<td>$c k \frac{z(x)^{k-1}}{B[1+z(x)]^{c}}$</td>
</tr>
<tr>
<td>Log-Normal $[0, \infty]$</td>
<td>$\frac{1}{Bx\sqrt{2\pi}} e^{-\frac{(\ln x - A)^2}{2B^2}}$</td>
<td>$\Phi\left(\frac{\ln x - A}{B}\right)$</td>
<td>$\frac{1}{B\sqrt{2\pi}}\Phi\left(\frac{\ln x - A}{B}\right) e^{-\frac{(\ln x - A)^2}{2B^2}}$</td>
</tr>
<tr>
<td>Student’s $t$ $[0, \infty]$</td>
<td>$\frac{2\Gamma\left(\frac{1+n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)B\sqrt{\pi n}}$</td>
<td>$*$</td>
<td>$*$</td>
</tr>
<tr>
<td>Cauchy $[0, \infty]$</td>
<td>$\frac{2}{\pi B[1+z(x)]^{2}}$</td>
<td>$\frac{2\arctan z(x)}{\pi}$</td>
<td>$\frac{B[1+z(x)]^{x}}{\frac{\pi}{2} - \arctan z(x)}$</td>
</tr>
<tr>
<td>F $[0, \infty]$</td>
<td>$\frac{\sqrt{B} \exp{-2z(x)}^{-1}}{(x-A)^{3/2}}$</td>
<td>$\text{erfc}\left(\sqrt{[2z(x)]^{-1}}\right)$</td>
<td>$\sqrt{\frac{B}{\pi}} \text{erf}\left(\frac{-2z(x)\exp{-2z(x)}^{-1}}{(x-A)^{3/2}}\right)$</td>
</tr>
<tr>
<td>Lévy $[A, \infty]$</td>
<td>$\sqrt{\frac{B}{\pi}} \exp{-2z(x)}^{-1}$</td>
<td>$\text{erfc}\left(\sqrt{[2z(x)]^{-1}}\right)$</td>
<td>$\sqrt{\frac{B}{\pi}} \text{erf}\left(\frac{-2z(x)\exp{-2z(x)}^{-1}}{(x-A)^{3/2}}\right)$</td>
</tr>
<tr>
<td>Log-Logistic $[0, \infty]$</td>
<td>$\text{This is a Burr distribution with } c=1$</td>
<td>$\text{This is a Chi distribution with } k=3$</td>
<td>$\text{This is a Weibull distribution with } k=2$</td>
</tr>
<tr>
<td>Maxwell $[0, \infty]$</td>
<td>$\text{This is a Weibull distribution with } k=2$</td>
<td>$\text{This is a Burr distribution with } c=1$</td>
<td>$\text{This is a Chi distribution with } k=3$</td>
</tr>
<tr>
<td>Arcsine $[0, 1]$</td>
<td>$\text{This is a Beta distribution with } \alpha = \beta = 1/2$</td>
<td>$\text{This is a Beta distribution with } \alpha = \beta = 1/2$</td>
<td>$\text{This is a Beta distribution with } \alpha = \beta = 1/2$</td>
</tr>
</tbody>
</table>

**Table 1:** List of common continuous IGFR distributions (standardized)
The Laplace distribution: The distribution is logconcave and therefore IFR.

The Weibull distribution: The distribution is logconcave and therefore IFR for \( k \geq 1 \). It is logconvex for \( 0 \leq k < 1 \), however, \((\log f(e^x))'' = -k^2e^{kx} \leq 0\), implying that the distribution is IGFR for \( k \geq 0 \).

The Gumbel (both Max and Min) distributions: Both distributions belong to the family of Extreme Value distributions. Both are logconcave and therefore IFR. In his note, Paul (2005) incorrectly states that the Gumbel Max distribution is neither IGFR nor DGFR. As the Appendix notes, this assertion is not true.

The Gamma distribution: The distribution is logconcave and therefore IFR for \( k \geq 1 \). It is logconvex for \( 0 \leq k < 1 \), however \((\log f(e^x))'' = -e^x < 0\), implying that the distribution is IGFR for \( k \geq 0 \). In the definition of the Gamma distribution, \( \Gamma(x) = \int_0^\infty t^{x-1}e^{-t} \, dt \) refers to the Gamma function.

The Beta distribution: The distribution is logconcave and therefore IFR for \( \alpha, \beta \geq 1 \). In addition, \((\log f(e^x))'' = -e^x(\beta - 1)(e^x - 1)^{-2}\), implying that the distribution is IGFR for \( \alpha > 0 \). Also, \((xf(x))' = (1 - x)^{\beta - 2}x^{\alpha - 1}[\alpha(1 - x) + x(1 - \beta)]/B(\alpha, \beta) \geq 0\) for \( \beta < 1 \), therefore (Lariviere(2006), Corollary 1) the Beta distribution is IGFR for any \( \alpha, \beta > 0 \). In the definition of the Beta distribution, \( B(\alpha, \beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha + \beta) \) refers to the Euler Beta function.

The Power Function distribution: The distribution is logconcave and therefore IFR for \( k \geq 1 \). It is logconvex for \( 0 \leq k < 1 \), however \((\log f(e^x))'' = 0\), implying that the distribution is IGFR for \( k \geq 0 \).

The Chi-squared distribution: The distribution is logconcave and therefore IFR when the number of degrees of freedom \( \nu \geq 2 \). However, \((\log f(e^x))'' = -e^x/2 \leq 0\), implying that the distribution is IGFR for \( \nu = 1 \).

The Chi distribution: The distribution is logconcave and therefore IFR when the number of degrees of freedom \( \nu \geq 1 \). However, \((\log f(e^x))'' = -2e^{2x} \leq 0\), implying that the distribution is IGFR for \( \nu \geq 0 \).

The Pareto distribution (also known as Pareto distribution of the first kind): The distribution is logconvex and monotone increasing and decreasing. However, \( g(x) = k \) so the distribution is both IGFR and DGFR for the shape parameter \( k \geq 0 \). The Pareto distribution also admits
a location parameter $A$ and a scale parameter $B$, in which case the distribution is known as a Pareto distribution of the second kind (or Lomax distribution). The density function is given by $f(x) = (k/B)[1 + (x - A)/B]^{-(k+1)}$. In this case, $g'(x) = k(B - A)(x + B - A)^{-2}$, and therefore the Pareto distribution of the second kind is IGFR if $A < B$, DGFR when $A > B$, and both IGFR and DGFR when $A = B$, (in particular for the standard Pareto distribution of the first kind, which has $A = B = 1$). Similar arguments can be made for the Pareto distributions of the third and fourth kind, respectively (see Johnson et al. (1994) for a detailed description of the Pareto family of distributions).

**The Burr distribution:** The standard Burr distribution is neither logconcave nor logconvex. However, $(\log f(e^x))'' = -c^2(k + 1)e^{cx}(1 + e^{cx})^{-2} < 0$, implying that the distribution is IGFR for $c, k > 0$. In general, once location and scale parameters are introduced, the Burr distribution is not necessarily IGFR. For example, when $k = 1$, the Burr distribution becomes the Pareto distribution of the second kind, which as we show above is DGFR whenever $A > B$. One can also show that $\lim_{x \to \infty} g(x) = ck$, which implies that if $g(x) > ck$ for some finite $x$, then the distribution is neither IGFR nor DGFR (e.g. when $c = 2, k = 2$ and $A = 1$ the maximum is attained at $\approx 4.8$, but $g(x)$ converges asymptotically to 4).

**The Log-Normal distribution:** The distribution is logconcave on $(0, 1)$ and logconvex on $(1, \infty)$. Since the logarithmic transform produces a Normal distribution, which is IFR, it follows that the Log-Normal family is IGFR.

**The Student’s $t$-distribution:** The distribution is logconcave on $[-\sqrt{v}, \sqrt{v}]$ and logconvex outside this interval. However, $(\log f(e^x))'' = -2v(v + 1)e^{2x}(v + e^{2x})^{-2} \leq 0$, implying that the distribution is IGFR for $v \geq 0$. If the distribution is generalized to include a location parameter $A$ and a scale parameter $B$, then this three parameter Student’s $t$-distribution is IGFR only when $A < 0$.

**The Cauchy distribution:** The distribution is a particular case of Student’s $t$-distribution with one degree of freedom. Therefore, it is IGFR. It is also a stable distribution, which means that the sum of two independent Cauchy random variables is Cauchy distributed. This implies that the sum of two Cauchy variables is also IGFR.

**The $F$ distribution:** The distribution is neither logconcave nor logconvex. However,

$$(\log f(e^x))'' = -v_1v_2(v_1 + v_2)e^x(v_2 + v_1e^x)^{-2}/2 \leq 0$$
implying that the distribution is IGFR for $\nu_1, \nu_2 \geq 0$.

The Lévy distribution: The distribution is logconcave on $(0, 2/3)$ and logconvex on $(2/3, \infty)$. However, $(\log f(e^x))'' = -e^{-x}/2 \leq 0$, implying that the distribution is IGFR. It is also a stable distribution, so the sum of two standard independent Lévy variables is also IGFR. If the Lévy distribution is parameterized with both location and scale parameters (say $A$ and $B$, respectively), then additional conditions are needed in order for the IGFR property to hold. For example, one such condition is $A \leq -1/3, 0 < B \leq -3A$. In the definition of the Lévy distribution, $\text{erfc}(x) = (2/\sqrt{\pi}) \int_x^\infty e^{-t^2} dt$ refers to the complementary error function.

3.2 Remarks on specific discrete distributions

The Geometric distribution is both logconcave and logconvex, and the negative binomial distribution can be either depending on its parameter $r$. All other distributions below are logconcave and therefore IGFR.

The Bernoulli distribution: The distribution has two outcomes: (i) failure ($X = 0$) with probability $1 - p$, and (ii) success ($X = 1$) with probability $p$.

The Discrete Uniform distribution: For this distribution, the probability of each outcome $x$ is $p$.

The Binomial distribution: The distribution models the sum of $n$ Bernoulli trials.

The Poisson distribution: The distribution models the number of occurrences per unit of time assuming an average rate of $\lambda$.

The Geometric distribution: For a sequence of Bernoulli trials, this distribution models the number of “failures” $x$ before the first “success”.

The Hypergeometric distribution: The distribution models the probability of observing $x$ “successes” in a total of $n$ trials that occur in a total population of size $N$, where the probability of a success is $m/N$. The support for this distribution is $\{\max\{0, n + m - N\}, \ldots, \min\{m, n\}\}$.

The Negative Binomial distribution: The distribution models the number of “successes” in a sequence of Bernoulli trials before $r$ “failures”. Here, $I_x(a, b) = \frac{\int_0^x t^{a-1}(1-t)^{b-1} dt}{B(a, b)} = \frac{B(x; a, b)}{B(a, b)}$ refers to the incomplete regularized Beta function. The distribution is IFR for $r > 1$, both IFR and DFR
Table 2: List of common discrete IGFR distributions

<table>
<thead>
<tr>
<th>Distribution</th>
<th>$P(X = x)$</th>
<th>$P(X \leq x)$</th>
<th>$g_X(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bernoulli {0, 1}</td>
<td>$\begin{cases} 1-p &amp; x = 0 \ p &amp; x = 1 \end{cases}$</td>
<td>$\begin{cases} 1-p &amp; 0 &lt; x &lt; 1 \ 1 &amp; x \geq 1 \end{cases}$</td>
<td>$\begin{cases} 0 &amp; x = 0 \ p &amp; x = 1 \end{cases}$</td>
</tr>
<tr>
<td>Discrete Uniform {0, 1, ..., 1/p}</td>
<td>$p$</td>
<td>$(x+1)p$</td>
<td>$\frac{xp}{1-xp}$</td>
</tr>
<tr>
<td>Binomial {0, 1, ..., n}</td>
<td>$\binom{n}{x} p^x (1-p)^{n-x}$</td>
<td>$\sum_{k=0}^{x} \binom{n}{k} p^k (1-p)^{n-k}$</td>
<td>$\frac{n}{\sum_{k=x}^{n} \binom{n}{k} p^k (1-p)^{n-k}}$</td>
</tr>
<tr>
<td>Poisson {0, 1, ...}</td>
<td>$\frac{\lambda^x}{x!} e^{-\lambda}$</td>
<td>$e^{-\lambda} \sum_{k=0}^{x} \frac{\lambda^k}{k!}$</td>
<td>$\frac{x \lambda^x}{x! \sum_{k=x}^{\infty} \frac{\lambda^k}{k!}}$</td>
</tr>
<tr>
<td>Geometric {0, 1, ...}</td>
<td>$(1-p)^x p$</td>
<td>$1 - (1-p)^{x+1}$</td>
<td>$xp$</td>
</tr>
<tr>
<td>Hypergeometric</td>
<td>$\binom{m}{x} \binom{N-m}{n-x} \binom{N}{n}$</td>
<td>$\sum_{k=0}^{x} \binom{m}{k} \binom{N-m}{n-k} \binom{N}{n}$</td>
<td>$\frac{x \binom{m}{x} \binom{N-m}{n-x} \binom{N}{n}}{\sum_{k=x}^{\min(m, n)} \binom{m}{k} \binom{N-m}{n-k} \binom{N}{n}}$</td>
</tr>
<tr>
<td>Negative Binomial ($r \geq 1$) {0, 1, ...}</td>
<td>$\binom{x + r - 1}{x} p^x (1-p)^r$</td>
<td>$1 - I_p(x + 1, r)$</td>
<td>$x \left( \frac{x + r - 1}{x} \right) p^x (1-p)^r \frac{1}{I_p(x, x)}$</td>
</tr>
</tbody>
</table>

{see notes}
for \( r = 1 \), and DFR for \( 0 < r < 1 \) (Barlow et al. 1963). However it is still IGFR since

\[
\Delta g = g_X(x + 1) - g_X(x) = (1 - p)^r p^x \left[ \frac{p(1 + x)}{(r + x)B(p; 1 + x, r)} - \frac{x}{(r + x - 1)B(p; x, r)} \right] \geq 0, \forall x \geq 0
\]

(signing the bracketed term follows from the basic properties of the incomplete Beta function).

\section*{Appendix: Corrections to Previous Results in the Literature}

\subsection*{IGFR Preservation Under Convolution}

Paul (2005) uses a counter example to show that IGFR distributions are not closed under ad-
dition by convolving a random variable \( X \) with increasing linear density and another random
variable \( Y \) with decreasing linear density, both with support on \([0, 2]\). (While not mentioned
in the proof, Paul (2005) assumes that \( X \) and \( Y \) are independent.) He claims that the result-
ing distribution is not IGFR since the generalized failure rate of the convolution is increasing
on the interval \([0, 2]\) and decreasing on \((2, 4]\). This assertion is incorrect. The generalized failure
rate is actually increasing on \([0, 4]\). (It is the density function of \( X + Y \) that is increasing
on \([0, 2]\) and then decreasing on \((2, 4]\).) To verify that the counter example does not hold, let
\( f_X(x) = 1 - x/2 \), \( f_Y(y) = y/2 \) with both variables defined over \([0, 2]\). With \( Z \) defined as \( X + Y \), we
get

\[
f_Z(z) = \int_0^z f_X(z - \xi) f_Y(\xi) d\xi = \frac{1}{2} \int_0^z f_X(z - \xi) \xi d\xi.
\]

Thus, we have

\[
f_Z(z) = \frac{1}{2} \int_0^z \left( 1 - \frac{z - \xi}{2} \right) \xi d\xi
\]

for \( 0 \leq z \leq 2 \) and

\[
f_Z(z) = \frac{1}{2} \int_{z-2}^2 \left( 1 - \frac{z - \xi}{2} \right) \xi d\xi
\]

for \( 2 < z \leq 4 \).

Integrating, we get,

\[
\begin{align*}
    f_Z(z) & = \begin{cases} 
    \frac{z^2(6-z)}{24} & 0 \leq z \leq 2 \\
    \frac{(z-4)^2(z+2)}{24} & 2 < z \leq 4
    \end{cases} \\
    F_Z(z) & = \begin{cases} 
    \frac{z^3(8-z)}{96} & 0 \leq z \leq 2 \\
    \frac{(z-2)^4-8(3z^2-20z+10)}{96} - 1 & 2 < z \leq 4
    \end{cases}
\end{align*}
\]

Figure 3 plots the failure rate and the generalized failure rate functions for \( Z \).

An indirect method of demonstrating the result above is to recognize that both the chosen linear increasing and decreasing densities are scaled versions, respectively, of the Beta(1, 2) and the Beta(2, 1) distributions. Since the Beta distribution is logconcave when both parameters are at
least 1, and the logconcavity property is preserved under positive scaling (Bagnoli and Bergstrom 2005), the respective convolution must also be logconcave, and therefore IGFR.

This error does not refute Paul’s general conjecture that IGFR distributions are not closed under convolution. Note that since IFR random variables are closed under addition (Barlow and Proschan 1965), at least one of the convolved variables must not be IFR for us to show that the IGFR property is not closed under convolution. One can verify analytically that the generalized failure rate of the convolution of two Log-Logistic distributions is not IGFR (note that the Log-Logistic distribution is IGFR). Let $X$ and $Y$ be two standard Log-Logistic random variables with $k = 2$ (see Table 3 below for the density function). $X$ and $Y$ are both IGFR (Figure 4, dashed line), but are neither IFR nor DFR, as the corresponding failure rate function is increasing in $[0, 1]$ and decreasing in $(1, \infty)$ (Figure 4, solid line).

The convolution $Z = X + Y$ is given by:

$$f_Z(z) = \int_0^\infty f_X(\xi) f_Y(z - \xi) d\xi = 4 \int_0^\infty \frac{\xi(z - \xi)}{(1 + \xi^2)^2 [1 + (z - \xi)^2]^2} d\xi$$

$$= \frac{4[z^2(z^4 - 16 + z(3z^2 - 4) \text{arctan } z) + 2(3z^4 + 6z^2 + 8) \log (1 + z^2)]}{z^3(4 + z^2)^3}$$

Figure 3: Convolution of linear increasing and decreasing random variables maintains the IFR and IGFR properties.
Integrating again, we obtain

\[ F_Z(z) = \int_0^z f_Z(\xi) d\xi = \frac{z^2[z^4 + 6z^2 - 4z \arctan z + 8] - 2(3z^2 + 4) \log (1 + z^2)}{z^2(4 + z^2)^2} \]

Substituting, the generalized failure rate is

\[ g_Z(z) = zh_Z(z) = \frac{2z^2[z^4 + z(3z^2 - 4) \arctan z - 16] + 4(3z^4 + 6z^2 + 8) \log (1 + z^2)}{(z^2 + 4)[z^2(z^2 + 2z \arctan z + 4) + (3z^2 + 4) \log (1 + z^2)]} \]

We have \( \lim_{z \to \infty} g_Z(z) = 2, g_Z(0^+) = 0 \), and the function also has a global maximum that is greater than 2 at \( z \approx 8.9 \), therefore it is neither monotone increasing nor monotone decreasing on the support of \( Z \). Figure 5 depicts the plot of the generalized failure rate function of the convolution. Hence, in general, it is indeed true that the IGFR property is not preserved under convolution. The next section discusses a sufficient condition that guarantees the preservation of the IGFR property under convolution. Section 4 lists a number of distributions for which convolution preserves the IGFR property.
Specific Distributions

Separately, Paul (2005) has two incorrect statements that relate to the prevalence of the IGFR property in the following two distributions.

**Generalized Logistic Distribution.** For the Generalized Logistic Distribution (see Section 3.1), we can verify that 
\[(\log(F'(x)))'' = -(C+1)B^{-2}\exp\{(x+A)/B\}\{\exp\{A/B\}+\exp\{x/B\}\}^{-2} \leq 0,\]
implying that the generalized Logistic distribution is log-concave, and thus IFR. In his note, Paul (2005) incorrectly claims that this distribution is neither IGFR nor DGFR.

**Gumbel Max Distribution** A second error in Paul (2005) relates to the Gumbel Max distribution. He states that the Gumbel Max distribution is neither IGFR nor DGFR. This assertion is not true. For a general Gumbel Max distribution with location parameter A and scale parameter B, the distribution function is given by 
\[F(x) = \exp\{-\exp\{(A - x)/B\}\}, \quad B > 0.\]
In this case, 
\[(\log(F'(x)))'' = -B^{-2}\exp\{(A - x)/B\} \leq 0,\]
implying the log-concavity of the distribution.

References


