

# On Dowling Geometries of Infinite Groups

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## Abstract

A finite subgeometry of a Dowling geometry for an infinite group is exhibited, which cannot be embedded in a Dowling geometry for any finite group; this provides a negative answer to a question of Bonin.

The question addressed herein is motivated by the following fundamental result of Rado, concerning the embeddability of finite geometries in projective geometries [Ra, Theorem 4]:

**Theorem.** *Every finite geometry representable over a field is representable over a finite field.*

In this short note we consider a corresponding embeddability question for a class of geometries known as *Dowling geometries*. We introduce only the essential concepts and definitions, and refer the reader to [BBB, Do, KK] for a more complete description of Dowling geometries and their properties.

Let  $n \geq 3$  be an integer, and let  $A$  be a group. The points of the Dowling geometry  $Q_n(A)$  are of two types: the *joints*  $p_1, p_2, \dots, p_n$ , which form a basis; and the *internal points*  $\alpha_{ij}$  ( $1 \leq i < j \leq n$ ), where  $\alpha$  denotes an element of  $A$  (we shall say that the point  $\alpha_{ij}$  is “labelled” by the group element  $\alpha$ ). There are also two types of nontrivial lines: the *coordinate lines*,  $l_{ij} = \{p_i, p_j\} \cup \{\alpha_{ij} \mid \alpha \in A\}$  ( $1 \leq i < j \leq n$ ); and the *transversal lines*,  $\{\alpha_{ij}, \beta_{jk}, (\alpha\beta)_{ik}\}$  ( $1 \leq i < j < k \leq n$ ), each containing three points, where  $\alpha\beta$  denotes the usual product in  $A$ . (The *trivial lines* are those containing only two points.)

In [Bo], Bonin posed the following question concerning these geometries:

**Question.** *If a finite geometry  $M$  embeds in a Dowling geometry  $Q_n(A)$  for some  $n \geq 3$  and infinite group  $A$ , can one always find a finite group  $B$  such that  $M$  embeds in  $Q_n(B)$ ?*

Bonin points to a number of striking similarities between Dowling and projective geometries, and suggests that we may view the former as group-theoretic analogues of the latter. In view of Rado’s result, an affirmative answer to Bonin’s question would further strengthen this analogy. We shall demonstrate here that, although this is not the case, it is a purely group theoretic phenomenon that ultimately settles the matter.

We will exhibit a particular infinite group  $A_0$  and finite subgeometry  $M_0$  of the Dowling geometry  $Q_3(A_0)$ , which cannot be embedded in  $Q_3(B)$  for *any* finite group  $B$ . We remark that, although our example is constructed for the single case  $n = 3$ , it is not a difficult matter to extend the construction to higher ranks. We also remark that the answer to Bonin's question is "yes" whenever the infinite group  $A$  is abelian; this was first observed by R. T. Tugger (cf. [Bo]).

From this point onward then we assume that  $n = 3$ . Here we may simplify our notation somewhat, denoting the transversal line  $\{\alpha_{12}, \beta_{23}, (\alpha\beta)_{13}\}$  simply by the triple  $(\alpha, \beta, \alpha\beta)$ . Observe that for  $\alpha, \beta, \gamma \in A$ , the triple  $(\alpha, \beta, \gamma)$  is a transversal line of  $Q_3(A)$  if and only if  $\alpha\beta = \gamma$ ; thus we have the following 1-1 correspondence:

$$\{ \text{transversal lines in } Q_3(A) \} \longleftrightarrow \{ \text{equations of the form } \alpha\beta = \gamma \text{ for } \alpha, \beta, \gamma \in A \}.$$

**Constructing  $M_0$ .** We now fix a particular infinite group introduced in a 1951 paper of Higman [Hi], namely the finitely presented group

$$A_0 = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid \alpha_1^{-1}\alpha_2\alpha_1 = \alpha_2^2, \alpha_2^{-1}\alpha_3\alpha_2 = \alpha_3^2, \alpha_3^{-1}\alpha_4\alpha_3 = \alpha_4^2, \alpha_4^{-1}\alpha_1\alpha_4 = \alpha_1^2 \rangle. \quad (1)$$

Higman's group is chosen here only as a concise example of a finitely presented infinite group having no nontrivial finite quotient group; there is nothing otherwise notable about our selection. For  $1 \leq k \leq 4$ , let  $\beta_k = \alpha_k^2$ , let  $\gamma_1 = \alpha_2\alpha_1$ ,  $\gamma_2 = \alpha_3\alpha_2$ ,  $\gamma_3 = \alpha_4\alpha_3$  and  $\gamma_4 = \alpha_1\alpha_4$ . Let  $\varepsilon$  denote the identity element of  $A_0$ , and consider the subset

$$\mathcal{S} := \{ \varepsilon, \alpha_k, \beta_k, \gamma_k \mid 1 \leq k \leq 4 \} \quad (2)$$

of elements of  $A_0$ . One can verify that the elements listed in (2) are distinct: for example, one can use the computer algebra system GAP [GAP] to verify by brute force that if one adjoins any relation of the form  $\delta_1 = \delta_2$  to the set of defining relations for  $A_0$  (where  $\delta_1, \delta_2$  are distinct elements of  $\mathcal{S}$ ), one obtains a presentation of the trivial group. We now define  $M_0$  to be the finite subgeometry of  $Q_3(A_0)$  obtained by restriction to the set

$$\Pi := \{ p_1, p_2, p_3 \} \cup \{ \delta_{ij} \mid \delta \in \mathcal{S}, 1 \leq i < j \leq 3 \}.$$

Note that  $M_0$  contains three "coordinate lines" (namely, the restrictions of the three coordinate lines  $l_{ij}$  of  $Q_3(A_0)$  to  $\Pi$ ), and also the following three families of transversal lines:  $\mathcal{L}_1 := \{ (\varepsilon, \delta, \delta) \mid \delta \in \mathcal{S} \}$ ,  $\mathcal{L}_2 := \{ (\delta, \varepsilon, \delta) \mid \delta \in \mathcal{S} \}$ , and

$$\mathcal{L}_3 := \left\{ \begin{array}{cccc} (\alpha_1, \alpha_1, \beta_1), & (\alpha_2, \alpha_2, \beta_2), & (\alpha_3, \alpha_3, \beta_3), & (\alpha_4, \alpha_4, \beta_4), \\ (\alpha_1, \alpha_4, \gamma_4), & (\alpha_2, \alpha_1, \gamma_1), & (\alpha_3, \alpha_2, \gamma_2), & (\alpha_4, \alpha_3, \gamma_3), \\ (\alpha_4, \beta_1, \gamma_4), & (\alpha_1, \beta_2, \gamma_1), & (\alpha_2, \beta_3, \gamma_2), & (\alpha_3, \beta_4, \gamma_3) \end{array} \right\}. \quad (3)$$

The lines in  $\mathcal{L}_1$  and  $\mathcal{L}_2$  clearly exist within  $Q_3(A_0)$ , as do those on the first two rows of (3). That the lines in the third row of (3) exist within  $Q_3(A_0)$  follows from the defining relations of  $A_0$  in (1). For example, the relation  $\alpha_4^{-1}\alpha_1\alpha_4 = \alpha_1^2$  is equivalent to the condition  $\alpha_1\alpha_4 = \alpha_4\alpha_1^2$ , which translates to the equation  $\alpha_4\beta_1 = \gamma_4$ . In fact, one can verify (again using **GAP** in much the same manner as earlier) that the union  $\mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$  contains *all* 3-point lines of  $M_0$ .

**Embedding  $M_0$ .** Consider any embedding  $f: M_0 \rightarrow Q_3(B)$  of  $M_0$  into the Dowling geometry of an arbitrary group  $B$ . Label and index the coordinate lines of  $Q_3(B)$  in such a way that  $l_{ij}$  maps, under  $f$ , to the coordinate line  $l'_{ij}$ . Then each transversal line  $\{\alpha_{12}, \beta_{23}, \gamma_{13}\}$  of  $M_0$  maps to the transversal line  $\{f(\alpha_{12}), f(\beta_{23}), f(\gamma_{13})\}$  of  $Q_3(B)$ . Furthermore, following  $f$  by an automorphism of the geometry  $Q_3(B)$ , we may assume that  $f(\varepsilon_{12})$  and  $f(\varepsilon_{23})$  are both labelled by  $e$ , the identity element of  $B$  (cf. [KK, section 7]); that is to say,  $f(\varepsilon_{12}) = e_{12} \in l'_{12}$  and  $f(\varepsilon_{23}) = e_{23} \in l'_{23}$ .

Next, for each  $\delta \in \mathcal{S}$ , let  $d \in B$  denote the label of  $f(\delta_{23})$  in  $l'_{23}$  (i.e.  $f(\delta_{23}) = d_{23}$ ). We claim that the labels for all of the remaining points in  $f(M_0)$  are now determined. Indeed, for each  $\delta \in \mathcal{S}$ , since the line  $(\varepsilon, \delta, \delta)$  in  $\mathcal{L}_1$  is mapped under  $f$  to the line  $(e, d, d)$  in  $Q_3(B)$ , it follows that  $d$  is also the label of  $f(\delta_{13})$  in  $l'_{13}$ . Similarly, images of lines in  $\mathcal{L}_2$  determine the labels of all of the points on  $l'_{12}$ . In particular,  $f$  induces a well-defined map  $\mathcal{S} \rightarrow B$ , sending

$$\alpha_k \mapsto a_k, \quad \beta_k \mapsto b_k, \quad \gamma_k \mapsto c_k \quad (1 \leq k \leq 4),$$

where the  $a_k$ ,  $b_k$  and  $c_k$  are distinct elements of  $B$ . More importantly, each line in  $\mathcal{L}_3$  determines an equation among elements of  $\mathcal{S}$  that must now hold among the corresponding elements of  $B$ ; these equations are as follows:

$$\begin{aligned} a_1^2 = b_1 & & a_2^2 = b_2 & & a_3^2 = b_3 & & a_4^2 = b_4 \\ a_1a_4 = c_4 & & a_2a_1 = c_1 & & a_3a_2 = c_2 & & a_4a_3 = c_3 \\ a_4b_1 = c_4 & & a_1b_2 = c_1 & & a_2b_3 = c_2 & & a_3b_4 = c_3 \end{aligned} \tag{4}$$

Evidently, this system of equations is equivalent to the following set of relations in  $B$ :

$$a_1^{-1}a_2a_1 = a_2^2, \quad a_2^{-1}a_3a_2 = a_3^2, \quad a_3^{-1}a_4a_3 = a_4^2, \quad a_4^{-1}a_1a_4 = a_1^2. \tag{5}$$

Of course these are precisely the relations satisfied by the generators of  $A_0$ , as given in the presentation (1). However, Higman also showed in [Hi] that the relations in (5) cannot hold among distinct elements of *any* finite group. It follows that  $M_0$  cannot embed in  $Q_3(B)$  for any finite group  $B$ .

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