Embeddability in Dowling geometries

(A report on recent joint work with

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Projective Geometries

Theorem [Rado, 1956] Every finite geometry representable over a field is representable over a finite field.

geometry means simple matroid (i.e. the empty set, and all singleton sets, are flats of the matroid.)

representable over a field means isomorphic to a vector matroid over a field.

Dowling Geometries

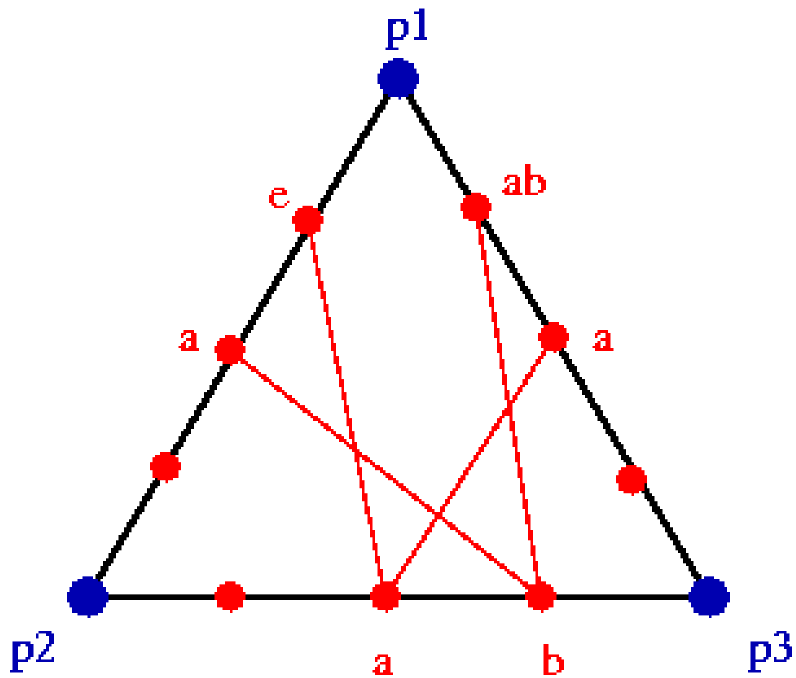
Let $r \geq 3$ be an integer, and let A be a group. The ground set of the geometry $Q_r(A)$ contains points of two flavours:

- joints: $p_1, \ldots, p_r$, which form a basis;
- internal points: α_{ij}, where $1 \leq i < j \leq r$, and $\alpha \in A$.

There are also two types of nontrivial lines:

- coordinate lines: $\{p_i, p_j\} \cup \{\alpha_{ij} \mid \alpha \in A\}$ $(1 \leq i < j \leq r)$;
- 3-point lines: $\{\alpha_{ij}, \beta_{jk}, (\alpha\beta)_{ik} \mid \alpha, \beta \in A\}$ $(1 \leq i < j < k \leq r)$.

The rank 3 case



Here we have a 1-1 correspondence:

$$\left\{ \begin{array}{c} \text{3-point lines} \\ \text{in } Q_3(A) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{equations of the form} \\ \alpha\beta = \gamma \text{ in } A \end{array} \right\}$$

Question

Recently, J. Bonin posed the following question, analogous to Rado's result for projective geometries:

Suppose it is known that a finite geometry embeds in a Dowling geometry $Q_r(A)$, for some $r \geq 3$ and group A. If desired, can A be assumed finite?

That is to say, can every finite subgeometry of $Q_r(A)$, where A is infinite, be embedded in some $Q_r(B)$, where B is finite?

The Subgeometries $M(\Gamma)$

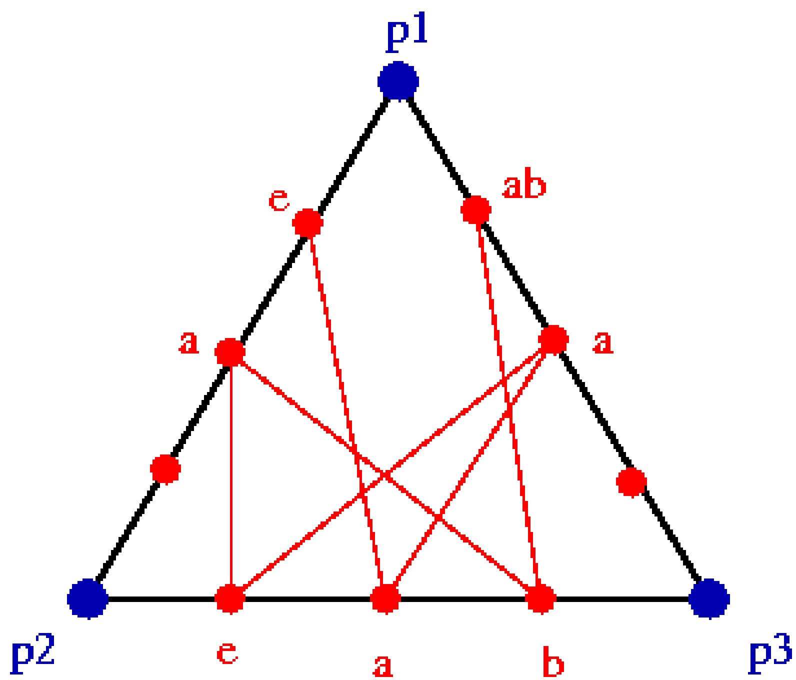
Consider a (rank 3) Dowling geometry $Q_3(A)$, and let Γ be a system of equations of the form $\alpha\beta = \gamma$ in A.

- Take $S \subset A$ to be the union of all elements of A appearing in some equation of Γ, together with the identity element ε of A.
- Put $E := \{\alpha_{ij} \mid 1 \leq i < j \leq 3, \alpha \in S\}$.
- Define a subgeometry $M = M(\Gamma)$ of $Q_3(A)$ by restriction to E.

In particular, note that M contains all 3-point lines $\{\alpha, \beta, \gamma\}$, where $\alpha\beta = \gamma$ is an equation in the system Γ.

Embedding $M(\Gamma)$

Let $M = M(\Gamma)$, and let $f : M \to Q_3(B)$ be an embedding into the rank 3 Dowling geometry of some group B. We may assume that $f(\varepsilon_{12}) = e_{12}$, and $f(\varepsilon_{23}) = e_{23}$, where e is the identity of B.



1. f induces a bijection from $S \subset A$ to some $T \subset B$.
2. Equations from Γ holding among elements of S must also hold among the corresponding elements of T.

Partial Cayley Tables

A system Γ of equations of the form $\alpha\beta = \gamma$ in A can also be thought of in terms of a partial Cayley table.

In view of observation 2, an affirmative answer to Bonin's question would require the following group-theoretic result to hold:

Every finite partial Cayley table of a group can be found as a partial Cayley table of a finite group.

Finitely Presented Groups

Every finite group presentation may be encoded as a partial Cayley table. For example, the set of defining relations for the group

$$A_0 = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \mid \alpha_2^{\alpha_1} = \alpha_2^2,\ \alpha_3^{\alpha_2} = \alpha_3^2,\ \alpha_4^{\alpha_3} = \alpha_4^2,\ \alpha_1^{\alpha_4} = \alpha_1^2 \rangle$$

is equivalent to the partial Cayley table

$$\Gamma = \left\{ \begin{array}{cccc} \alpha_1^2 = \beta_1 & \alpha_2^2 = \beta_2 & \alpha_3^2 = \beta_3 & \alpha_4^2 = \beta_4 \\ \alpha_1\alpha_4 = \gamma_4 & \alpha_2\alpha_1 = \gamma_1 & \alpha_3\alpha_2 = \gamma_2 & \alpha_4\alpha_3 = \gamma_3 \\ \alpha_4\beta_1 = \gamma_4 & \alpha_1\beta_2 = \gamma_1 & \alpha_2\beta_3 = \gamma_2 & \alpha_3\beta_4 = \gamma_3 \end{array} \right\}$$

on the twelve elements $\{\alpha_i, \beta_i, \gamma_i \mid 1 \le i \le 4\}$.

Conclusion

In 1953, G. Higman showed that the group A_0 is infinite, and also that it has no finite quotient groups. More particularly, he demonstrated that the defining relations of A_0 cannot hold among nontrivial elements in any finite group.

Let $M = M(\Gamma)$ be the subgeometry of $Q_3(A_0)$ obtained from the system Γ equivalent to the defining relations of A_0. Then M cannot be embedded in $Q_3(B)$, for any finite group B.

This answers Bonin's question.