Algorithms for algebras with involution

Peter Brooksbank

Bucknell University

Group Theory, Combinatorics and Computation

The University of Western Australia, Perth

5–16 January, 2009

In honour of Professor Cheryl Praeger's 60th Birthday

The principal motivation is a solution to the following problem:

IsometryGroup

Given: A system $\{b_1, b_2, \ldots, b_n\}$ of classical forms on a finite vector space V.

Construct: Generators for the group of linear transformations that preserve all of the forms in the system.

Equivalently:

Given an hermitian bilinear map $b\colon V \times V \to W$, construct its isometry group:

$$\mathrm{Isom}(b) = \{g \in \mathrm{GL}(V) \colon b(ug, vg) = b(u, v) \text{ for all } u, v \in V\}$$

The principal motivation is a solution to the following problem:

IsometryGroup

Given: A system $\{b_1, b_2, \ldots, b_n\}$ of classical forms on a finite vector space V.

Construct: Generators for the group of linear transformations that preserve all of the forms in the system.

Equivalently:

Given an hermitian bilinear map $b\colon V \times V \to W$, construct its isometry group:

$$\mathrm{Isom}(b) = \{g \in \mathrm{GL}(V) : b(ug, vg) = b(u, v) \text{ for all } u, v \in V\}$$

The principal motivation is a solution to the following problem:

IsometryGroup

Given: A system $\{b_1, b_2, \ldots, b_n\}$ of classical forms on a finite vector space V.

Construct: Generators for the group of linear transformations that preserve all of the forms in the system.

Equivalently:

Given an hermitian bilinear map $b\colon V \times V \to W$, construct its isometry group:

$$\mathrm{Isom}(b) = \{g \in \mathrm{GL}(V) \colon b(ug, vg) = b(u, v) \text{ for all } u, v \in V\}$$

The principal motivation is a solution to the following problem:

IsometryGroup

Given: A system $\{b_1, b_2, \ldots, b_n\}$ of classical forms on a finite vector space V.

Construct: Generators for the group of linear transformations that preserve all of the forms in the system.

Equivalently:

Given an hermitian bilinear map $b\colon V \times V \to W$, construct its isometry group:

$$\mathrm{Isom}(b) = \{g \in \mathrm{GL}(V) : b(ug, vg) = b(u, v) \text{ for all } u, v \in V\}$$

The principal motivation is a solution to the following problem:

IsometryGroup

Given: A system $\{b_1, b_2, \ldots, b_n\}$ of classical forms on a finite vector space V.

Construct: Generators for the group of linear transformations that preserve all of the forms in the system.

Equivalently:

Given an hermitian bilinear map $b \colon V \times V \to W$, construct its isometry group:

$$\mathrm{Isom}(b) = \{g \in \mathrm{GL}(V) \colon b(ug, vg) = b(u, v) \text{ for all } u, v \in V\}$$

The principal motivation is a solution to the following problem:

IsometryGroup

Given: A system $\{b_1, b_2, \ldots, b_n\}$ of classical forms on a finite vector space V.

Construct: Generators for the group of linear transformations that preserve all of the forms in the system.

Equivalently:

Given an hermitian bilinear map $b\colon V \times V \to W$, construct its isometry group:

$$\mathrm{Isom}(b) = \{g \in \mathrm{GL}(V)\colon b(ug, vg) = b(u, v) \text{ for all } u, v \in V\}$$

Structure of (finite) associative algebras

Let $A \leqslant M_d(F)$ be an algebra over a finite field F. Then

$$A \;=\; J(A) \;\oplus\; (S_1 \oplus S_2 \oplus \ldots \oplus S_t),$$

where $J(A)$ is the Jacobson radical of A. Furthermore,

$$S_j \cong M_{e_j}(K_j),$$

where K_j is a finite extension field of F.

Structure of (finite) associative algebras

Let $A \leqslant M_d(F)$ be an algebra over a finite field F. Then

$$A \ = \ J(A) \ \oplus \ (S_1 \oplus S_2 \oplus \ldots \oplus S_t),$$

where $J(A)$ is the Jacobson radical of A. Furthermore,

$$S_j \cong M_{e_j}(K_j),$$

where K_j is a finite extension field of F.

Structure of (finite) associative algebras

Let $A \leqslant M_d(F)$ be an algebra over a finite field F. Then

$$A = J(A) \oplus (S_1 \oplus S_2 \oplus \ldots \oplus S_t),$$

where $J(A)$ is the Jacobson radical of A. Furthermore,

$$S_j \cong M_{e_j}(K_j),$$

where K_j is a finite extension field of F.

Polynomial-time theory of finite algebras

A is given as the enveloping algebra of some $X \subset M_d(F)$:

$$A := \mathrm{Env}(X) = F\text{-span of } \overline{X} \text{ (semigroup generated by } X)$$

There are polynomial time, Las Vegas algorithms to solve:

WedderburnDecomposition [Ivanyos]

Given $A \leqslant M_d(F)$, find generators for the Jacobson radical $J(A)$ and for a subring S of A such that $A = J(A) \oplus S$.

DecomposeSemisimple [Eberly & Giesbrecht]

1. Given $S \leqslant M_d(F)$ semisimple, find ideals $S_1, \ldots, S_t$ of S such that $S = S_1 \oplus \ldots \oplus S_t$.
2. For each $1 \leqslant j \leqslant t$, find an isomorphism $S_j \to M_{e_j}(K_j)$.

Polynomial-time theory of finite algebras

A is given as the enveloping algebra of some $X \subset M_d(F)$:

$$A := \mathrm{Env}(X) = F\text{-span of } \overline{X} \text{ (semigroup generated by } X\text{)}$$

There are polynomial time, Las Vegas algorithms to solve:

WedderburnDecomposition [Ivanyos]

Given $A \leqslant M_d(F)$, find generators for the Jacobson radical $J(A)$ and for a subring S of A such that $A = J(A) \oplus S$.

DecomposeSemisimple [Eberly & Giesbrecht]

1. Given $S \leqslant M_d(F)$ semisimple, find ideals $S_1, \ldots, S_t$ of S such that $S = S_1 \oplus \ldots \oplus S_t$.
2. For each $1 \leqslant j \leqslant t$, find an isomorphism $S_j \to M_{e_j}(K_j)$.

Polynomial-time theory of finite algebras

A is given as the enveloping algebra of some $X \subset M_d(F)$:

$$A := \mathrm{Env}(X) = F\text{-span of } \overline{X} \text{ (semigroup generated by } X)$$

There are polynomial time, Las Vegas algorithms to solve:

WedderburnDecomposition [Ivanyos]

Given $A \leqslant M_d(F)$, find generators for the Jacobson radical $J(A)$ and for a subring S of A such that $A = J(A) \oplus S$.

DecomposeSemisimple [Eberly & Giesbrecht]

1. Given $S \leqslant M_d(F)$ semisimple, find ideals $S_1, \ldots, S_t$ of S such that $S = S_1 \oplus \ldots \oplus S_t$.
2. For each $1 \leqslant j \leqslant t$, find an isomorphism $S_j \to M_{e_j}(K_j)$.

Polynomial-time theory of finite algebras

A is given as the enveloping algebra of some $X \subset M_d(F)$:

$$A := \mathrm{Env}(X) = F\text{-span of } \overline{X} \text{ (semigroup generated by } X)$$

There are polynomial time, Las Vegas algorithms to solve:

`WedderburnDecomposition` [Ivanyos]

Given $A \leqslant M_d(F)$, find generators for the Jacobson radical $J(A)$ and for a subring S of A such that $A = J(A) \oplus S$.

`DecomposeSemisimple` [Eberly & Giesbrecht]

1. Given $S \leqslant M_d(F)$ semisimple, find ideals $S_1, \ldots, S_t$ of S such that $S = S_1 \oplus \ldots \oplus S_t$.
2. For each $1 \leqslant j \leqslant t$, find an isomorphism $S_j \to M_{e_j}(K_j)$.

∗-algebra

$A \leqslant M_d(F)$ equipped with an involutory anti-automorphism:

- $(a^*)^* = a$
- $(a+b)^* = a^* + b^*$
- $(ab)^* = b^* a^*$

for all $a, b \in A$.

Algebra of adjoints of a bilinear map

Let b be a nondegenerate hermitian bilinear map $V \times V \to W$.
The algebra of adjoints of b is defined to be

$$\mathrm{Adj}(b) = \{(f, g) \in \mathrm{End}(V) \times \mathrm{End}(V)^{\mathrm{op}} : b(uf, v) = b(u, vg)\ \forall u, v \in V\}$$

1. $(f, g) \in \mathrm{Adj}(b) \iff (g, f) \in \mathrm{Adj}(b)$
2. g is uniquely determined by f.
3. $f^* := g$ is an involution on $\mathrm{Adj}(b)$

Algebra of adjoints of a bilinear map

Let b be a nondegenerate hermitian bilinear map $V \times V \to W$.
The algebra of adjoints of b is defined to be

$$\mathrm{Adj}(b) = \{(f, g) \in \mathrm{End}(V) \times \mathrm{End}(V)^{\mathrm{op}} : b(uf, v) = b(u, vg)\ \forall u, v \in V\}$$

1. $(f, g) \in \mathrm{Adj}(b) \iff (g, f) \in \mathrm{Adj}(b)$
2. g is uniquely determined by f.
3. $f^* := g$ is an involution on $\mathrm{Adj}(b)$

Algebra of adjoints of a bilinear map

Let b be a nondegenerate hermitian bilinear map $V \times V \to W$.
The algebra of adjoints of b is defined to be

$$\mathrm{Adj}(b) = \{(f, g) \in \mathrm{End}(V) \times \mathrm{End}(V)^{\mathrm{op}} : b(uf, v) = b(u, vg)\ \forall u, v \in V\}$$

1. $(f, g) \in \mathrm{Adj}(b) \iff (g, f) \in \mathrm{Adj}(b)$
2. g is uniquely determined by f.
3. $f^* := g$ is an involution on $\mathrm{Adj}(b)$

Algebra of adjoints of a bilinear map

Let b be a nondegenerate hermitian bilinear map $V \times V \to W$. The **algebra of adjoints** of b is defined to be

$$\mathrm{Adj}(b) = \{(f, g) \in \mathrm{End}(V) \times \mathrm{End}(V)^{\mathrm{op}} : b(uf, v) = b(u, vg)\ \forall u, v \in V\}$$

1. $(f, g) \in \mathrm{Adj}(b) \iff (g, f) \in \mathrm{Adj}(b)$
2. g is uniquely determined by f.
3. $f^* := g$ is an involution on $\mathrm{Adj}(b)$

Algebra of adjoints of a bilinear map

Let b be a nondegenerate hermitian bilinear map $V \times V \to W$.
The algebra of adjoints of b is defined to be

$$\mathrm{Adj}(b) = \{(f, g) \in \mathrm{End}(V) \times \mathrm{End}(V)^{\mathrm{op}} : b(uf, v) = b(u, vg)\ \forall u, v \in V\}$$

1. $(f, g) \in \mathrm{Adj}(b) \iff (g, f) \in \mathrm{Adj}(b)$
2. g is uniquely determined by f.
3. $f^* := g$ is an involution on $\mathrm{Adj}(b)$

Group algebras

Let G be a finite group.

Define an involution on on the group algebra FG as follows:

$$\left[\sum_{g\in G} \alpha_g g\right]^* := \sum_{g\in G} \alpha_g g^{-1}$$

Group algebras

Let G be a finite group.

Define an involution on on the group algebra FG as follows:

$$\left[\sum_{g\in G} \alpha_g g\right]^* := \sum_{g\in G} \alpha_g g^{-1}$$

Structure of ∗-algebras

- ∗-ideal: an ideal of A that is ∗-invariant.
- $J(A)$ is a ∗-ideal of A, so $A/J(A)$ is a ∗-algebra.
- [Taft] If $\mathrm{char}(F) \neq 2$ then there is a ∗-invariant subring complement to $J(A)$.
- ∗-simple ring: a ∗-ring having no proper ∗-ideals.
- $A/J(A) = S_1 \oplus \ldots \oplus S_t$ where each S_j is ∗-simple.

Structure of $*$-algebras

- $*$-ideal: an ideal of A that is $*$-invariant.
- $J(A)$ is a $*$-ideal of A, so $A/J(A)$ is a $*$-algebra.
- [Taft] If $\mathrm{char}(F) \neq 2$ then there is a $*$-invariant subring complement to $J(A)$.
- $*$-simple ring: a $*$-ring having no proper $*$-ideals.
- $A/J(A) = S_1 \oplus \ldots \oplus S_t$ where each S_j is $*$-simple.

Structure of $*$-algebras

- $*$-ideal: an ideal of A that is $*$-invariant.
- $J(A)$ is a $*$-ideal of A, so $A/J(A)$ is a $*$-algebra.
- [Taft] If $\mathrm{char}(F) \neq 2$ then there is a $*$-invariant subring complement to $J(A)$.
- $*$-simple ring: a $*$-ring having no proper $*$-ideals.
- $A/J(A) = S_1 \oplus \ldots \oplus S_t$ where each S_j is $*$-simple.

Structure of $*$-algebras

- $*$-ideal: an ideal of A that is $*$-invariant.
- $J(A)$ is a $*$-ideal of A, so $A/J(A)$ is a $*$-algebra.
- [Taft] If $\mathrm{char}(F) \neq 2$ then there is a $*$-invariant subring complement to $J(A)$.
- $*$-simple ring: a $*$-ring having no proper $*$-ideals.
- $A/J(A) = S_1 \oplus \ldots \oplus S_t$ where each S_j is $*$-simple.

Structure of $*$-algebras

- $*$-ideal: an ideal of A that is $*$-invariant.
- $J(A)$ is a $*$-ideal of A, so $A/J(A)$ is a $*$-algebra.
- [Taft] If $\mathrm{char}(F) \neq 2$ then there is a $*$-invariant subring complement to $J(A)$.
- $*$-simple ring: a $*$-ring having no proper $*$-ideals.
- $A/J(A) = S_1 \oplus \ldots \oplus S_t$ where each S_j is $*$-simple.

Structure of $*$-algebras

- $*$-ideal: an ideal of A that is $*$-invariant.
- $J(A)$ is a $*$-ideal of A, so $A/J(A)$ is a $*$-algebra.
- [Taft] If $\mathrm{char}(F) \neq 2$ then there is a $*$-invariant subring complement to $J(A)$.
- $*$-simple ring: a $*$-ring having no proper $*$-ideals.
- $A/J(A) = S_1 \oplus \ldots \oplus S_t$ where each S_j is $*$-simple.

$*$-simple rings

Let $S \leqslant M_d(F)$ be $*$-simple.

Then either S is simple as a ring, or $S = T_1 \oplus T_2$, where the T_i are isomorphic simple rings interchanged by $*$.

More precisely, S is $*$-isomorphic to one of the following:

1. $\mathrm{Adj}(d)$, where d is a classical form over an extension K.
2. $(M_e(K) \oplus M_e(K), \bullet)$ where $(x, y)^\bullet = (y, x)$.

∗-simple rings

Let $S \leqslant M_d(F)$ be ∗-simple.

Then either S is simple as a ring, or $S = T_1 \oplus T_2$, where the T_i are isomorphic simple rings interchanged by ∗.

More precisely, S is ∗-isomorphic to one of the following:

1. $\mathrm{Adj}(d)$, where d is a classical form over an extension K.
2. $(M_e(K) \oplus M_e(K), \bullet)$ where $(x, y)^\bullet = (y, x)$.

∗-simple rings

Let $S \leqslant M_d(F)$ be ∗-simple.

Then either S is simple as a ring, or $S = T_1 \oplus T_2$, where the T_i are isomorphic simple rings interchanged by ∗.

More precisely, S is ∗-isomorphic to one of the following:

1. $\mathrm{Adj}(d)$, where d is a classical form over an extension K.
2. $(M_e(K) \oplus M_e(K), \bullet)$ where $(x, y)^\bullet = (y, x)$.

∗-simple rings

Let $S \leqslant M_d(F)$ be ∗-simple.

Then either S is simple as a ring, or $S = T_1 \oplus T_2$, where the T_i are isomorphic simple rings interchanged by ∗.

More precisely, S is ∗-isomorphic to one of the following:

1. $\mathrm{Adj}(d)$, where d is a classical form over an extension K.
2. $(M_e(K) \oplus M_e(K), \bullet)$ where $(x, y)^\bullet = (y, x)$.

∗-simple rings

Let $S \leqslant M_d(F)$ be ∗-simple.

Then either S is simple as a ring, or $S = T_1 \oplus T_2$, where the T_i are isomorphic simple rings interchanged by ∗.

More precisely, S is ∗-isomorphic to one of the following:

1. $\mathrm{Adj}(d)$, where d is a classical form over an extension K.
2. $(M_e(K) \oplus M_e(K), \bullet)$ where $(x, y)^\bullet = (y, x)$.

The input

We assume that a $*$-algebra is specified as

$$A = \mathrm{Env}(X) \leqslant M_d(F)$$

together with a procedure that evaluates the involution on A.

Algorithmic complexity is measured in terms of the number of field operations performed, and on the number of invocations of the involution procedure on A.

The input

We assume that a $*$-algebra is specified as

$$A = \mathrm{Env}(X) \leqslant M_d(F)$$

together with a procedure that evaluates the involution on A.

Algorithmic complexity is measured in terms of the number of field operations performed, and on the number of invocations of the involution procedure on A.

Computing a $*$-invariant decomposition

The following algorithm is based on Taft's original proof.

`TaftDecomposition`(A) [char$(F) \neq 2$]

$J \oplus S \leftarrow$ `WedderburnDecomposition`(A)
$K \leftarrow J$
while $K \neq 0$ **do**
    /* $S = \mathrm{Env}(Y)$ and $\pi : A \to S$ */
    $B \leftarrow \mathrm{Env}(\{\frac{1}{2}(y + y^*\pi^*) : y \in Y\})$
    $K \oplus S \leftarrow$ `WedderburnDecomposition`(B)
return $A = J \oplus S$

Computing a $*$-invariant decomposition

The following algorithm is based on Taft's original proof.

`TaftDecomposition`(A) $[\mathrm{char}(F) \neq 2]$

$J \oplus S \leftarrow$ `WedderburnDecomposition`(A)
$K \leftarrow J$
while $K \neq 0$ **do**
  /* $S = \mathrm{Env}(Y)$ and $\pi : A \to S$ */
  $B \leftarrow \mathrm{Env}(\{\frac{1}{2}(y + y^*\pi^*) : y \in Y\})$
  $K \oplus S \leftarrow$ `WedderburnDecomposition`(B)
return $A = J \oplus S$

Constructive recognition of $*$-simple factors

Suppose that $S = \mathrm{Env}(X) \leqslant M_d(F)$ is $*$-simple of classical type.

[S is $*$-isomorphic to some $\mathrm{Adj}(d)$ where d is a classical form]

1. Assume S acts irreducibly on its underlying module.
2. Construct a $*$-isomorphism $S \to M_e(K)$.
   [Assume S acts absolutely irreducibly on its module]
3. We now seek a matrix $D \in M_e(K)$ such that

   $$s \cdot D = D \cdot s^{*\mathrm{tr}}$$

   for all $s \in S$. Construct an isomorphism of modules:

   $$\{x : x \in X\} \longrightarrow \{x^{*\mathrm{tr}} : x \in X\}$$

Constructive recognition of $*$-simple factors

Suppose that $S = \mathrm{Env}(X) \leqslant M_d(F)$ is $*$-simple of classical type.
[S is $*$-isomorphic to some $\mathrm{Adj}(d)$ where d is a classical form]

1. Assume S acts irreducibly on its underlying module.
2. Construct a $*$-isomorphism $S \to M_e(K)$.
   [Assume S acts absolutely irreducibly on its module]
3. We now seek a matrix $D \in M_e(K)$ such that

$$s \cdot D = D \cdot s^{*\mathrm{tr}}$$

for all $s \in S$. Construct an isomorphism of modules:

$$\{x : x \in X\} \longrightarrow \{x^{*\mathrm{tr}} : x \in X\}$$

Constructive recognition of $*$-simple factors

Suppose that $S = \mathrm{Env}(X) \leqslant M_d(F)$ is $*$-simple of classical type.
[S is $*$-isomorphic to some $\mathrm{Adj}(d)$ where d is a classical form]

1. Assume S acts irreducibly on its underlying module.
2. Construct a $*$-isomorphism $S \to M_e(K)$.
   [Assume S acts absolutely irreducibly on its module]
3. We now seek a matrix $D \in M_e(K)$ such that

$$s \cdot D = D \cdot s^{*\mathrm{tr}}$$

for all $s \in S$. Construct an isomorphism of modules:

$$\{x : x \in X\} \longrightarrow \{x^{*\mathrm{tr}} : x \in X\}$$

Constructive recognition of ∗-simple factors

Suppose that $S = \mathrm{Env}(X) \leqslant M_d(F)$ is ∗-simple of classical type.
[S is ∗-isomorphic to some $\mathrm{Adj}(d)$ where d is a classical form]

1. Assume S acts irreducibly on its underlying module.
2. Construct a ∗-isomorphism $S \to M_e(K)$.
   [Assume S acts absolutely irreducibly on its module]
3. We now seek a matrix $D \in M_e(K)$ such that

$$s \cdot D = D \cdot s^{*\mathrm{tr}}$$

for all $s \in S$. Construct an isomorphism of modules:

$$\{x : x \in X\} \longrightarrow \{x^{*\mathrm{tr}} : x \in X\}$$

Constructive recognition of $*$-simple factors

Suppose that $S = \mathrm{Env}(X) \leqslant M_d(F)$ is $*$-simple of classical type.
[S is $*$-isomorphic to some $\mathrm{Adj}(d)$ where d is a classical form]

1. Assume S acts irreducibly on its underlying module.
2. Construct a $*$-isomorphism $S \to M_e(K)$.
   [Assume S acts absolutely irreducibly on its module]
3. We now seek a matrix $D \in M_e(K)$ such that

$$s \cdot D = D \cdot s^{*\mathrm{tr}}$$

for all $s \in S$. Construct an isomorphism of modules:

$$\{x : x \in X\} \longrightarrow \{x^{*\mathrm{tr}} : x \in X\}$$

Constructive recognition of $*$-simple factors

Suppose that $S = \mathrm{Env}(X) \leqslant M_d(F)$ is $*$-simple of classical type.
[S is $*$-isomorphic to some $\mathrm{Adj}(d)$ where d is a classical form]

1. Assume S acts irreducibly on its underlying module.
2. Construct a $*$-isomorphism $S \to M_e(K)$.
   [Assume S acts absolutely irreducibly on its module]
3. We now seek a matrix $D \in M_e(K)$ such that

$$s \cdot D = D \cdot s^{*\mathrm{tr}}$$

for all $s \in S$. Construct an isomorphism of modules:

$$\{x : x \in X\} \longrightarrow \{x^{*\mathrm{tr}} : x \in X\}$$

Constructive recognition of $*$-simple factors

Suppose that $S = \mathrm{Env}(X) \leqslant M_d(F)$ is $*$-simple of classical type.
[S is $*$-isomorphic to some $\mathrm{Adj}(d)$ where d is a classical form]

1. Assume S acts irreducibly on its underlying module.
2. Construct a $*$-isomorphism $S \to M_e(K)$.
   [Assume S acts absolutely irreducibly on its module]
3. We now seek a matrix $D \in M_e(K)$ such that

$$s \cdot D = D \cdot s^{*\mathrm{tr}}$$

for all $s \in S$. Construct an isomorphism of modules:

$$\{x : x \in X\} \quad \longrightarrow \quad \{x^{*\mathrm{tr}} : x \in X\}$$

- We have a preliminary implementation in MAGMA.
- The code been used to compute the structure of $*$-algebras of dimension up to 200 over fields of moderate size.

- We have a preliminary implementation in MAGMA.
- The code been used to compute the structure of ∗-algebras of dimension up to 200 over fields of moderate size.

- We have a preliminary implementation in MAGMA.
- The code been used to compute the structure of $*$-algebras of dimension up to 200 over fields of moderate size.

Constructing the group preserving a system of forms

$\{b_1, \ldots, b_n\} \quad \leftrightarrow \quad$ bilinear map b

$$\begin{aligned} \mathrm{Isom}(b) &= \{g \in \mathrm{GL}(V) : b(ug, vg) = b(u, v) \text{ for all } u, v \in V\} \\ &= \{g \in \mathrm{GL}(V) : (g, g^{-1}) \in \mathrm{Adj}(b)\} \\ &= \{g \in \mathrm{Adj}(b) : gg^* = 1\} \end{aligned}$$

... $\mathrm{Isom}(b)$ is the norm group of the $*$-algebra $\mathrm{Adj}(b)$.

$$\begin{array}{ccccc} \mathrm{Adj}(b) & = & J(\mathrm{Adj}(b)) & \oplus & (\mathrm{Adj}(d_1) \oplus \ldots \oplus \mathrm{Adj}(d_t)) \\ & & \updownarrow & & \updownarrow \\ \mathrm{Isom}(b) & = & O_p(\mathrm{Isom}(b)) & \rtimes & (\mathrm{Isom}(d_1) \times \ldots \times \mathrm{Isom}(d_t)) \end{array}$$

Constructing the group preserving a system of forms

$$\{b_1, \ldots, b_n\} \quad \leftrightarrow \quad \text{bilinear map } b$$

$$\begin{aligned}
\mathrm{Isom}(b) &= \{g \in \mathrm{GL}(V) : b(ug, vg) = b(u, v) \text{ for all } u, v \in V\} \\
&= \{g \in \mathrm{GL}(V) : (g, g^{-1}) \in \mathrm{Adj}(b)\} \\
&= \{g \in \mathrm{Adj}(b) : gg^* = 1\}
\end{aligned}$$

... $\mathrm{Isom}(b)$ is the norm group of the $*$-algebra $\mathrm{Adj}(b)$.

$$\begin{array}{ccccc}
\mathrm{Adj}(b) & = & J(\mathrm{Adj}(b)) & \oplus & (\mathrm{Adj}(d_1) \oplus \ldots \oplus \mathrm{Adj}(d_t)) \\
 & & \updownarrow & & \updownarrow \\
\mathrm{Isom}(b) & = & O_p(\mathrm{Isom}(b)) & \rtimes & (\mathrm{Isom}(d_1) \times \ldots \times \mathrm{Isom}(d_t))
\end{array}$$

Constructing the group preserving a system of forms

$\{b_1, \ldots, b_n\} \quad \leftrightarrow \quad$ bilinear map b

$$\begin{aligned} \mathrm{Isom}(b) &= \{g \in \mathrm{GL}(V) \colon b(ug, vg) = b(u,v) \text{ for all } u, v \in V\} \\ &= \{g \in \mathrm{GL}(V) \colon (g, g^{-1}) \in \mathrm{Adj}(b)\} \\ &= \{g \in \mathrm{Adj}(b) \colon gg^* = 1\} \end{aligned}$$

... $\mathrm{Isom}(b)$ is the norm group of the ∗-algebra $\mathrm{Adj}(b)$.

$$\begin{array}{ccccc} \mathrm{Adj}(b) & = & J(\mathrm{Adj}(b)) & \oplus & (\mathrm{Adj}(d_1) \oplus \ldots \oplus \mathrm{Adj}(d_t)) \\ & & \updownarrow & & \updownarrow \\ \mathrm{Isom}(b) & = & O_p(\mathrm{Isom}(b)) & \rtimes & (\mathrm{Isom}(d_1) \times \ldots \times \mathrm{Isom}(d_t)) \end{array}$$

Constructing the group preserving a system of forms

$$\{b_1, \ldots, b_n\} \quad \leftrightarrow \quad \text{bilinear map } b$$

$$\begin{aligned} \mathrm{Isom}(b) &= \{g \in \mathrm{GL}(V) \colon b(ug, vg) = b(u, v) \text{ for all } u, v \in V\} \\ &= \{g \in \mathrm{GL}(V) \colon (g, g^{-1}) \in \mathrm{Adj}(b)\} \\ &= \{g \in \mathrm{Adj}(b) \colon gg^{*} = 1\} \end{aligned}$$

... $\mathrm{Isom}(b)$ is the norm group of the $*$-algebra $\mathrm{Adj}(b)$.

$$\begin{array}{ccccc} \mathrm{Adj}(b) & = & J(\mathrm{Adj}(b)) & \oplus & (\mathrm{Adj}(d_1) \oplus \ldots \oplus \mathrm{Adj}(d_t)) \\ & & \updownarrow & & \updownarrow \\ \mathrm{Isom}(b) & = & O_p(\mathrm{Isom}(b)) & \rtimes & (\mathrm{Isom}(d_1) \times \ldots \times \mathrm{Isom}(d_t)) \end{array}$$

Constructing the group preserving a system of forms

$$\{b_1,\dots,b_n\} \quad \leftrightarrow \quad \text{bilinear map } b$$

$$\begin{aligned}\mathrm{Isom}(b) &= \{g \in \mathrm{GL}(V) \colon b(ug, vg) = b(u,v) \text{ for all } u, v \in V\} \\ &= \{g \in \mathrm{GL}(V) \colon (g, g^{-1}) \in \mathrm{Adj}(b)\} \\ &= \{g \in \mathrm{Adj}(b) \colon gg^* = 1\}\end{aligned}$$

... $\mathrm{Isom}(b)$ is the norm group of the $*$-algebra $\mathrm{Adj}(b)$.

$$\begin{array}{ccccc} \mathrm{Adj}(b) & = & J(\mathrm{Adj}(b)) & \oplus & (\mathrm{Adj}(d_1) \oplus \dots \oplus \mathrm{Adj}(d_t)) \\ & & \updownarrow & & \updownarrow \\ \mathrm{Isom}(b) & = & O_p(\mathrm{Isom}(b)) & \rtimes & (\mathrm{Isom}(d_1) \times \dots \times \mathrm{Isom}(d_t)) \end{array}$$

Concluding remarks and a question

1. The construction of $\mathrm{Adj}(b)$ from the given b is the main bottleneck in practice.
2. We are currently developing similar ideas to construct the pseudo-isometry group of b.
(This group is the stabiliser of the subspace spanned by the system $\{b_1, \ldots, b_n\}$ used to define b.)
3. Are there interesting questions about group algebras that could possibly be investigated using these algorithms?

Happy Birthday Cheryl!

Concluding remarks and a question

1. The construction of $\mathrm{Adj}(b)$ from the given b is the main bottleneck in practice.
2. We are currently developing similar ideas to construct the pseudo-isometry group of b.
   (This group is the stabiliser of the subspace spanned by the system $\{b_1, \ldots, b_n\}$ used to define b.)
3. Are there interesting questions about group algebras that could possibly be investigated using these algorithms?

Happy Birthday Cheryl!

Concluding remarks and a question

1. The construction of $\mathrm{Adj}(b)$ from the given b is the main bottleneck in practice.
2. We are currently developing similar ideas to construct the pseudo-isometry group of b.
   (This group is the stabiliser of the subspace spanned by the system $\{b_1, \ldots, b_n\}$ used to define b.)
3. Are there interesting questions about group algebras that could possibly be investigated using these algorithms?

Happy Birthday Cheryl!

Concluding remarks and a question

1. The construction of $\mathrm{Adj}(b)$ from the given b is the main bottleneck in practice.
2. We are currently developing similar ideas to construct the pseudo-isometry group of b.
(This group is the stabiliser of the subspace spanned by the system $\{b_1, \ldots, b_n\}$ used to define b.)
3. Are there interesting questions about group algebras that could possibly be investigated using these algorithms?

Happy Birthday Cheryl!

Concluding remarks and a question

1. The construction of $\mathrm{Adj}(b)$ from the given b is the main bottleneck in practice.
2. We are currently developing similar ideas to construct the pseudo-isometry group of b.
   (This group is the stabiliser of the subspace spanned by the system $\{b_1, \ldots, b_n\}$ used to define b.)
3. Are there interesting questions about group algebras that could possibly be investigated using these algorithms?

Happy Birthday Cheryl!

Concluding remarks and a question

1. The construction of $\mathrm{Adj}(b)$ from the given b is the main bottleneck in practice.
2. We are currently developing similar ideas to construct the pseudo-isometry group of b.
(This group is the stabiliser of the subspace spanned by the system $\{b_1, \ldots, b_n\}$ used to define b.)
3. Are there interesting questions about group algebras that could possibly be investigated using these algorithms?

Happy Birthday Cheryl!