Permutation Edge-Labellings of Partially Ordered Sets

Peter McNamara

CMS Summer 2002 Meeting
17th June 2002

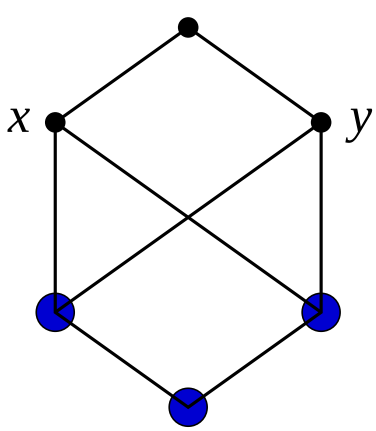
Slides and preprint available from

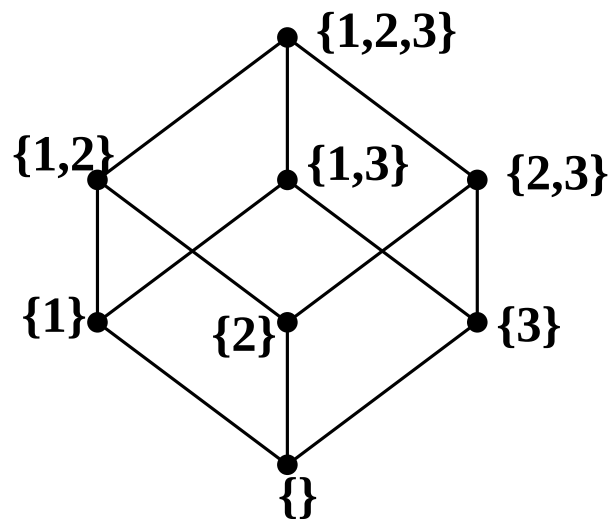
`http://www-math.mit.edu/~mcnamara/`

Definition A partially ordered set (poset) P is said to be a *lattice* if every two elements x and y of P have a least upper bound and a greatest lower bound.

We call the least upper bound the *join* of x and y and denote it by $x \vee y$.

We call the greatest lower bound the *meet* of x and y and denote it by $x \wedge y$.





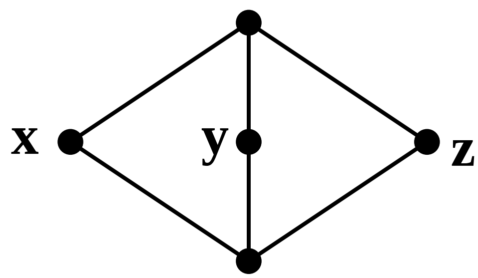
We say that a lattice L is *distributive* if

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

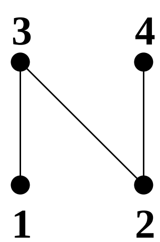
and

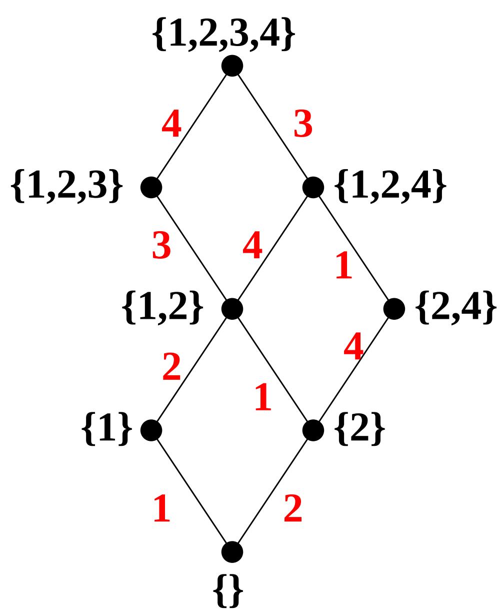
$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

for all elements x, y and z of L.



EXAMPLE The lattice of *order ideals* of a poset P.





An edge-labelling of a poset P is said to be an S_n *EL-labelling* if it satisfies the following 2 conditions:

1. Every interval $[x, y]$ of P has exactly one maximal chain with increasing labels
2. The labels along any maximal chain form a permutation of n.

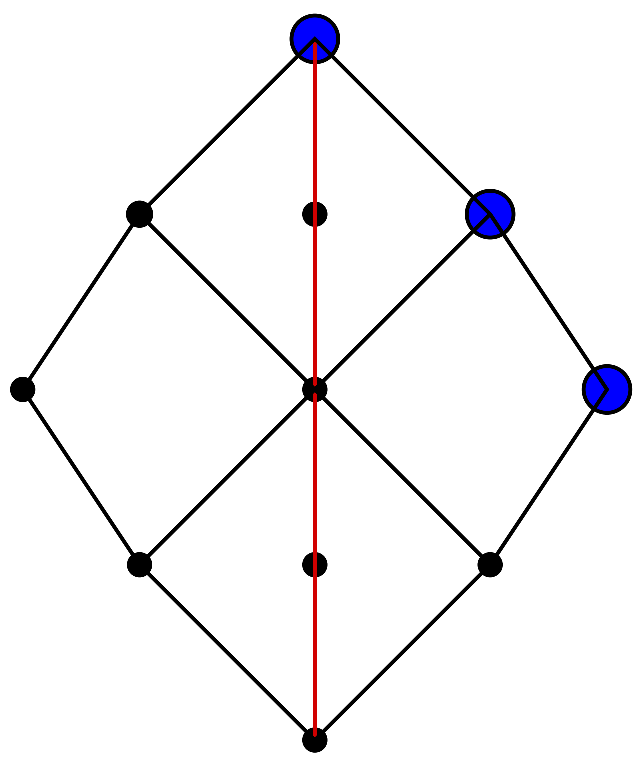
Who cares?

- EL-labelling $\Rightarrow$ Shellable $\Rightarrow$ Cohen-Macaulay

What other classes of posets have S_n EL-labellings?

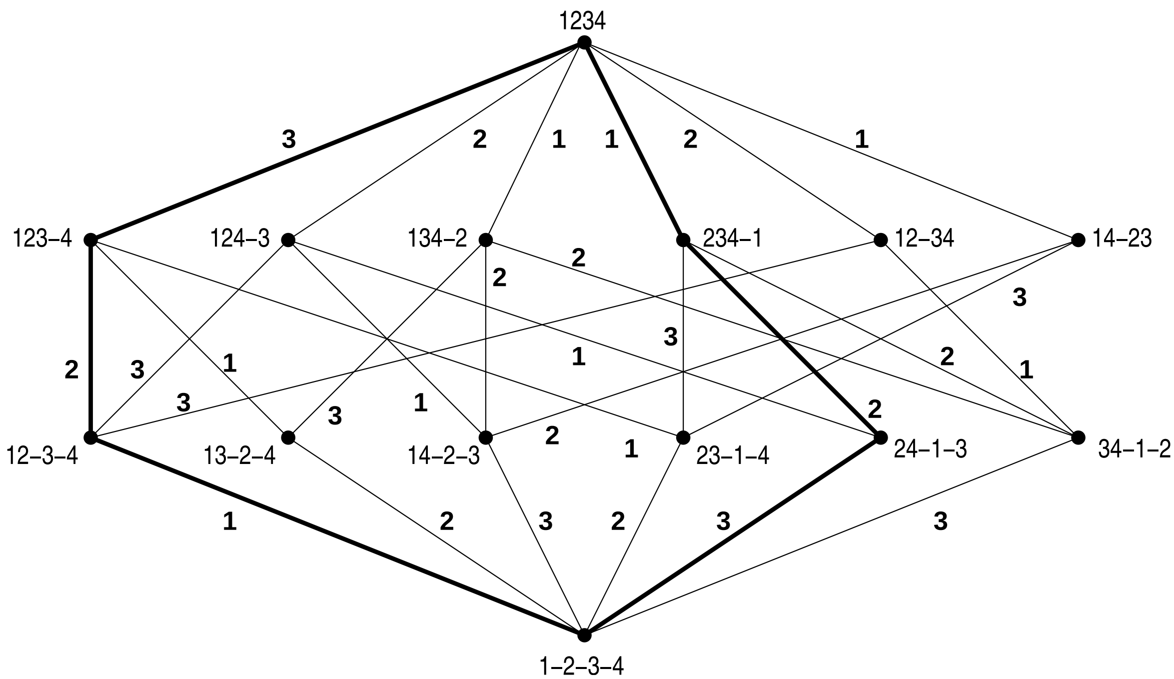
Definition A finite lattice L is said to be *supersolvable* if it contains a maximal chain $\mathfrak{m}$, called an *M-chain* of L, which together with any other chain of L generates a distributive sublattice.

EXAMPLE



QUESTION (R. Stanley) Are there any other lattices that have S_n EL-labellings?

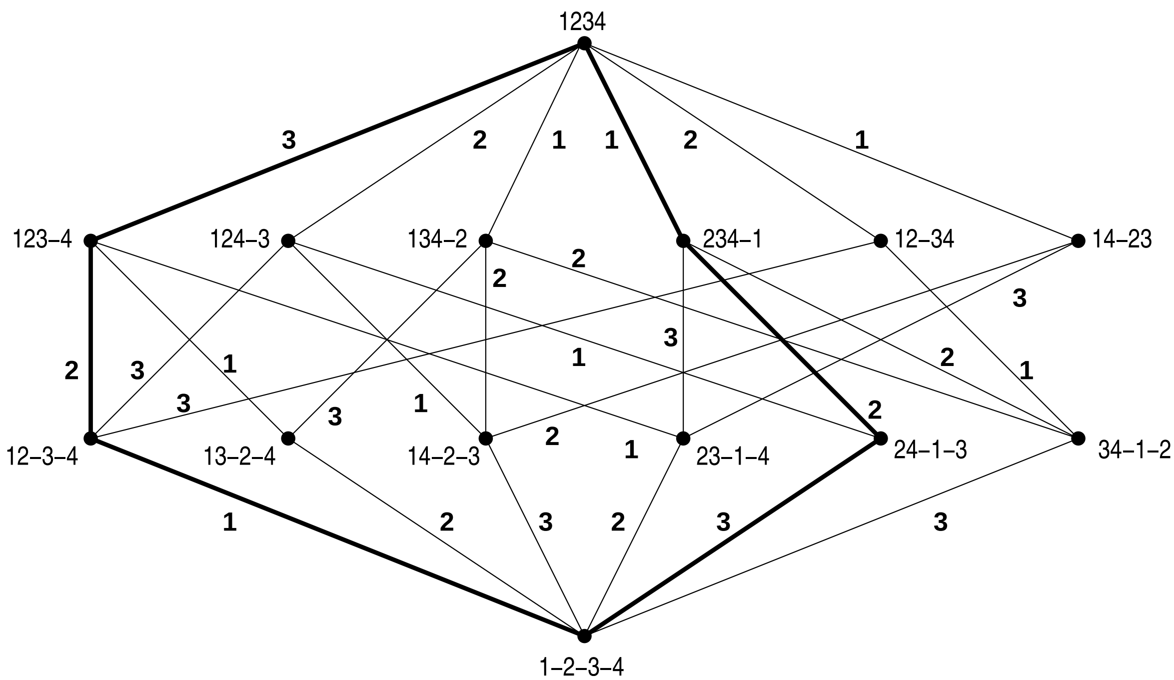
Theorem 1 *A finite lattice has an S_n EL-labelling if and only if it is supersolvable.*



We want the chain $\mathfrak{m}_0$ with labels $1, 2, 3, \ldots, n$ to be an M-chain. Let $\mathfrak{m}$ be any other chain of L. (It suffices to consider only maximal chains.) The proof relies on the equivalence of the following 2 posets:

1. The sublattice of L generated by $\mathfrak{m}$ and $\mathfrak{m}_0$

2. If $\mathfrak{m}$ has a *descent* at i, then we define $T_i(\mathfrak{m})$ to be the unique chain in L differing from $\mathfrak{m}$ only at level i and having no descent at i. If $\mathfrak{m}$ doesn't have a descent at i then we set $T_i(\mathfrak{m}) = \mathfrak{m}$. Then we take the "closure" of $\mathfrak{m}$ in L under the action of $T_1, T_2, \ldots, T_{n-1}$.



The action of $T_1, T_2, \ldots, T_{n-1}$ has the following properties:

1. It is a local action: it only changes a chain in at most one place
2. ${T_i}^2 = T_i$
3. $T_i T_j = T_j T_i$ if $|i-j| \geq 2$
4. $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$
5. $\mathrm{ch}(\chi(x)) = \omega(F_L(x))$

An action on the maximal chains of a lattice having all of these properties is called a *good $\mathcal{H}_n(0)$ action*.

Theorem 2 *A finite lattice has a good $\mathcal{H}_n(0)$ action if and only if it has an S_n EL-labeling.*

Corollary *Let L be a finite lattice. TFAE:*

1. *L is supersolvable*
2. *L has an S_n EL-labelling*
3. *L has a good $\mathcal{H}_n(0)$ action*