A Combinatorial Classification of Skew Schur Functions

Peter McNamara
Bucknell University

Joint work with Stephanie van Willigenburg

Workshop on Combinatorial Hopf Algebras and Macdonald Polynomials
CRM, 11 May 2007

Slides and paper available from
`www.facstaff.bucknell.edu/pm040/`

When are Two Skew Schur Functions Equal?

Peter McNamara

Bucknell University

Joint work with Stephanie van Willigenburg

Workshop on Combinatorial Hopf Algebras and Macdonald Polynomials
CRM, 11 May 2007

Slides and paper available from
`www.facstaff.bucknell.edu/pm040/`

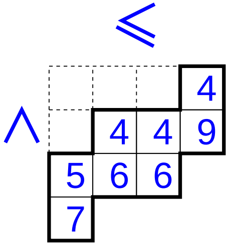
Outline

- Recent work on skew Schur function equality
- Skew Schur equivalence
- Composition of skew diagrams, main results
- Conjectures, open problems

Skew Schur functions

- English notation
- Infinite number of variables

Example:
$\lambda/\mu = (4,4,3,1)/(3,1)$

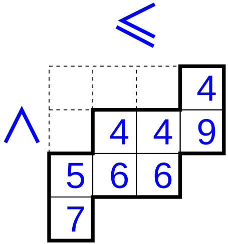


The skew Schur function $s_{\lambda/\mu}$ in the variables $x = (x_1, x_2, \ldots)$ is defined by

$$s_{\lambda/\mu} = \sum_{\text{SSYT } T} x_1^{\#\text{1's in } T} x_2^{\#\text{2's in } T} \cdots .$$

Skew Schur functions

- English notation
- Infinite number of variables



Example:
$\lambda/\mu = (4, 4, 3, 1)/(3, 1)$

The skew Schur function $s_{\lambda/\mu}$ in the variables $x = (x_1, x_2, \ldots)$ is defined by

$$s_{\lambda/\mu} = \sum_{\text{SSYT } T} x_1^{\#1\text{'s in } T} x_2^{\#2\text{'s in } T} \cdots .$$

- All the usual reasons we care about Schur functions and skew Schur functions:
  - They are symmetric functions
  - Schur functions form a basis
  - Connections with Algebraic Geometry, Rep. Theory, etc.

Big Question: When is $s_{\lambda/\alpha} = s_{\mu/\beta}$?

The HDL series

Big Question: When is $s_{\lambda/\alpha} = s_{\mu/\beta}$?

- Lou Billera, Hugh Thomas, Steph van Willigenburg (2004):

Big Question: When is $s_{\lambda/\alpha} = s_{\mu/\beta}$?

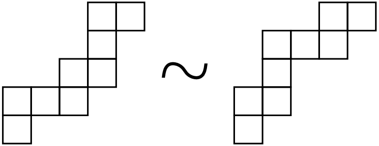
- Lou Billera, Hugh Thomas, Steph van Willigenburg (2004):

The HDL series

Big Question: When is $s_{\lambda/\alpha} = s_{\mu/\beta}$?

- Lou Billera, Hugh Thomas, Steph van Willigenburg (2004):

Complete classification of equality of ribbon Schur functions



- HDL II: Vic Reiner, Kristin Shaw, Steph van Willigenburg (2006):
  - The more general setting of binomial syzygies

    $$cs_{D_1} s_{D_2} \cdots s_{D_m} = c' s_{D'_1} s_{D'_2} \cdots s_{D'_n}$$

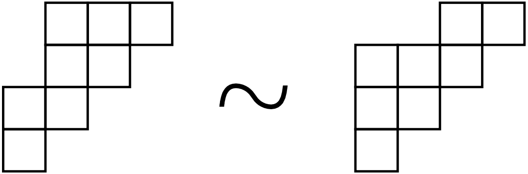
    is equivalent to understanding equalities among connected skew diagrams.
  - 3 operations for generating skew diagrams with equal skew Schur functions.
  - Necessary conditions, but of a different flavor.

The HDL series

- HDL III: McN., Steph van Willigenburg (2006):
  - An operation that encompasses the three operations of HDL II.
  - Theorem that generalizes all previous results.
    Explains the 6 missing equivalences from HDL II.
  - Conjecture for necessary and sufficient conditions for $s_{\lambda/\alpha} = s_{\mu/\beta}$.
    Reflects classification of HDL I for ribbons.

Skew diagrams (skew shapes) D, E.
If $s_D = s_E$, we will write $D \sim E$.

Example



We want to classify all equivalences classes, thereby classifying all skew Schur functions.

EC2, Exercise 7.56(a) [2-]

Theorem

$D \sim D^*$, where D^* denotes D rotated by 180°.

EC2, Exercise 7.56(a) [2-]

Theorem

$D \sim D^*$, where D^* denotes D rotated by 180°.

Goal: Use this equivalence to build other skew equivalences.

EC2, Exercise 7.56(a) [2-]

Theorem

$D \sim D^*$, where D^* denotes D rotated by 180°.

Goal: Use this equivalence to build other skew equivalences.

Where we're headed:

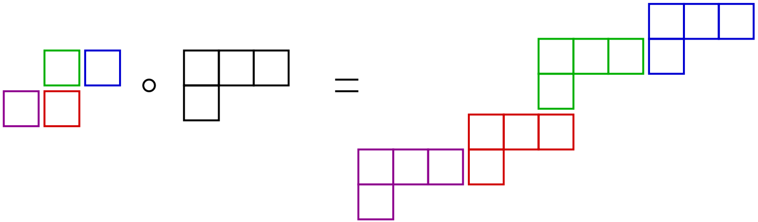
Theorem

Suppose we have skew diagrams D, D' and E satisfying certain assumptions. If $D \sim D'$ then

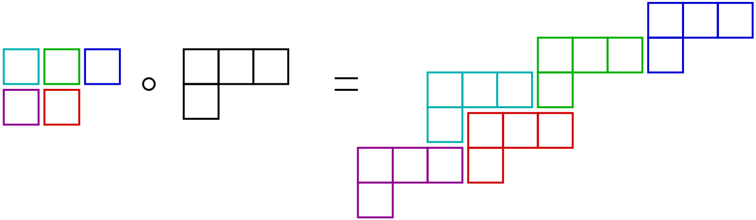
$$D' \circ_W E \quad \sim \quad D \circ_W E \quad \sim \quad D \circ_{W^*} E^*.$$

Main definition: composition of skew diagrams.

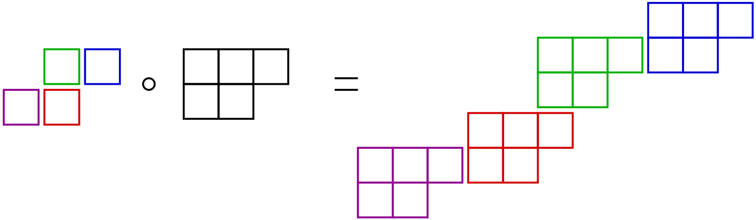
Composition of skew diagams



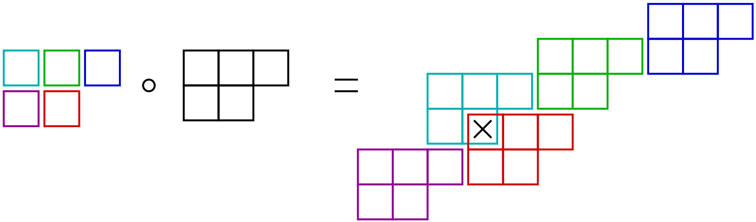
Composition of skew diagams



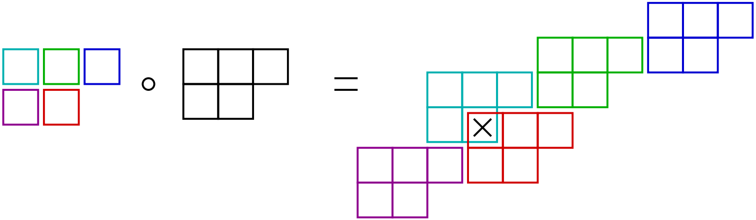
Composition of skew diagams



Composition of skew diagams

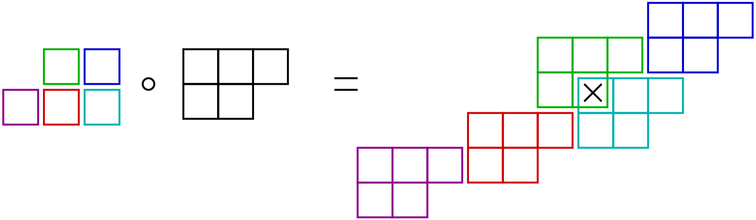


Composition of skew diagams



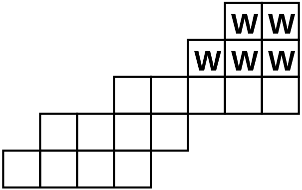
Theorem [McN., van Willigenburg] *If $D \sim D'$, then*

$$D' \circ E \quad \sim \quad D \circ E \quad \sim \quad D \circ E^*.$$



Actually, the previous slide was just a warm-up....

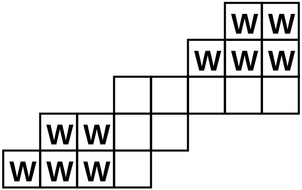
A skew diagram W *lies in the top* of a skew diagram E if W appears as a connected subdiagram of E that includes the northeasternmost cell of E.



Amalgamated Compositions

Actually, the previous slide was just a warm-up....

A skew diagram W *lies in the top* of a skew diagram E if W appears as a connected subdiagram of E that includes the northeasternmost cell of E.



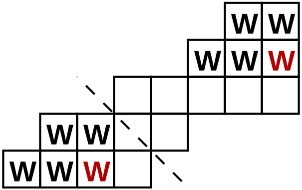
Similarly, W *lies in the bottom* of E.

Our interest: W lies in both the top and bottom of E. We write $E = WOW$.

Amalgamated Compositions

Actually, the previous slide was just a warm-up....

A skew diagram W *lies in the top* of a skew diagram E if W appears as a connected subdiagram of E that includes the northeasternmost cell of E.



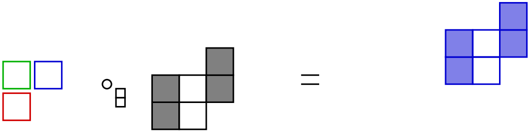
Similarly, W *lies in the bottom* of E.

Our interest: W lies in both the top and bottom of E. We write $E = WOW$.

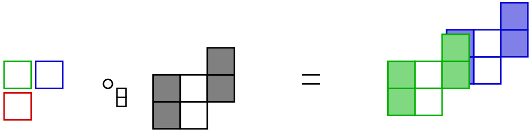
Hypotheses: (inspired by hypotheses of RSvW)

1. W_{ne} and W_{sw} are separated by at least one diagonal.
2. $E \setminus W_{ne}$ and $E \setminus W_{sw}$ are both connected skew diagrams.
3. W is maximal given its set of diagonals.

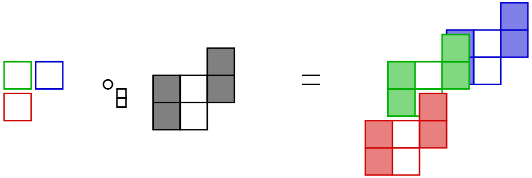
Amalgamated Compositions



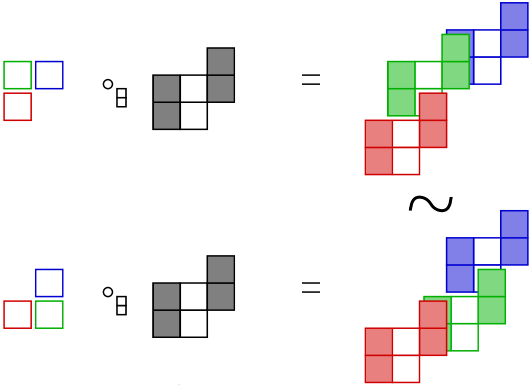
Amalgamated Compositions



Amalgamated Compositions

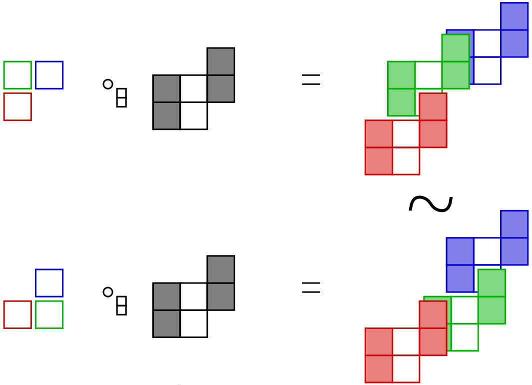


Amalgamated Compositions



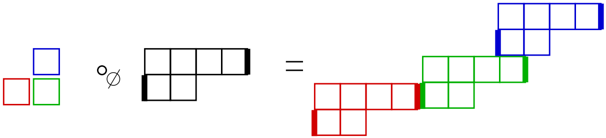
15 boxes: first of the non-RSvW examples

Amalgamated Compositions

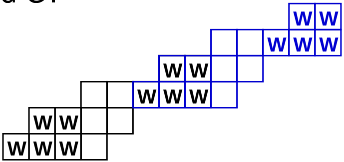


15 boxes: first of the non-RSvW examples

If $W = \emptyset$, we get the regular compositions:

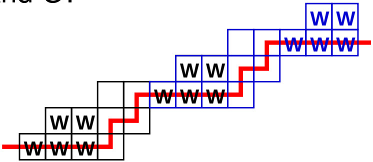


Construction of $\overline{W}$ and $\overline{O}$:



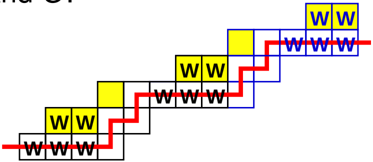
What are the results?

Construction of $\overline{W}$ and $\overline{O}$:



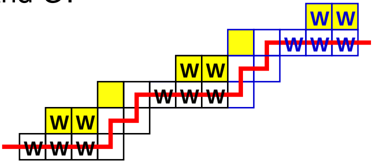
What are the results?

Construction of $\overline{W}$ and $\overline{O}$:



Hypothesis 4. $\overline{W}$ is never adjacent to $\overline{O}$.

Construction of $\overline{W}$ and $\overline{O}$:

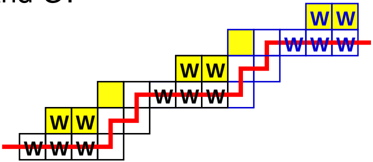


Hypothesis 4. $\overline{W}$ is never adjacent to $\overline{O}$.

Conjecture. Suppose we have skew diagrams D, D' with $D \sim D'$ and $E = WOW$ satisfying Hypotheses 1-4, then

$$D' \circ_W E \;\sim\; D \circ_W E \;\sim\; D \circ_{W^*} E^*.$$

Construction of $\overline{W}$ and $\overline{O}$:

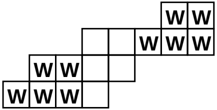


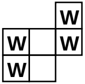
Hypothesis 4. $\overline{W}$ is never adjacent to $\overline{O}$.

Conjecture. Suppose we have skew diagrams D, D' with $D \sim D'$ and $E = WOW$ satisfying Hypotheses 1-4, then

$$D' \circ_W E \sim D \circ_W E \sim D \circ_{W^*} E^*.$$

Hypothesis 5. In $E = WOW$, at least one copy of W has just one cell adjacent to O.





What are the results?

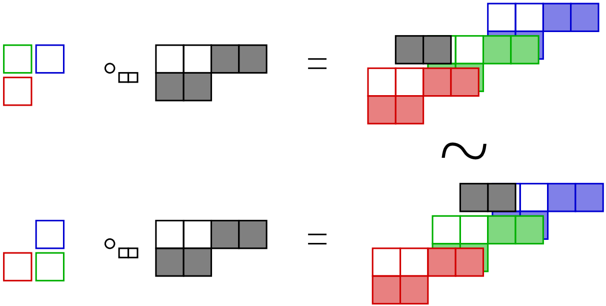
Theorem.[McN., van Willigenburg] Suppose we have skew diagrams D, D' with $D \sim D'$ and $E = WOW$ satisfying Hypotheses 1-5, then

$$D' \circ_W E \quad \sim \quad D \circ_W E \quad \sim \quad D \circ_{W^*} E^*.$$

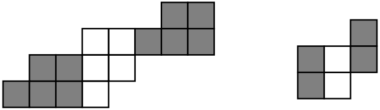
What are the results?

Theorem.[McN., van Willigenburg] Suppose we have skew diagrams D, D' with $D \sim D'$ and $E = WOW$ satisfying Hypotheses 1-5, then

$$D' \circ_W E \quad \sim \quad D \circ_W E \quad \sim \quad D \circ_{W^*} E^*.$$



15 boxes: second of the non-RSvW examples



A word or two about the proof

The hard part: An expression for $s_{D \circ_W E}$ in terms of s_D, s_E, $s_{\overline{W}}$, $s_{\overline{O}}$:

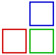
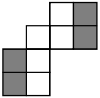
$$s_{D \circ_W E} \left(s_{\overline{W}} \right)^{|\widehat{D}|} \left(s_{\overline{O}} \right)^{|\widetilde{D}|} = \pm (s_D \circ_W s_E).$$

The easy part: The blue portion is invariant if we replace D by D' when $D' \sim D$. Similary, can replace E by E^*.

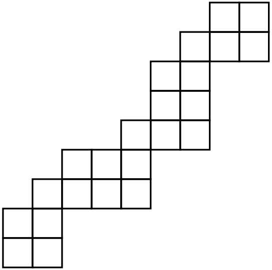
Proof of expression uses:

- Hamel-Goulden determinants. See also paper of Chen, Yan, Yang.
- Sylvester's Determinantal Identity.

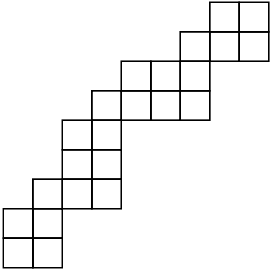
- Removing Hypothesis 5.

$D=$  $E=$ 

$D \circ_W E$ has 23 boxes, and $D \circ_W E \sim D^* \circ_W E$:



$\sim$



Main open problem

Theorem. [McN, van Willigenburg]
Skew diagrams $E_1, E_2, \ldots, E_r$
$E_i = W_i O_i W_i$ satisfies Hypotheses 1-5
E_i' and W_i' denote either E_i and W_i, or E_i^* and W_i^*.
Then

$$((\cdots(E_1 \circ_{W_2} E_2) \circ_{W_3} E_3) \cdots) \circ_{W_r} E_r \sim ((\cdots(E_1' \circ_{W_2'} E_2') \circ_{W_3'} E_3') \cdots) \circ_{W_r} E_r' .$$

Main open problem

Theorem. [McN, van Willigenburg]
Skew diagrams $E_1, E_2, \ldots, E_r$
$E_i = W_i O_i W_i$ satisfies Hypotheses 1-5
E_i' and W_i' denote either E_i and W_i, or E_i^* and W_i^*.
Then

$$((\cdots(E_1 \circ_{W_2} E_2) \circ_{W_3} E_3) \cdots) \circ_{W_r} E_r \sim ((\cdots(E_1' \circ_{W_2'} E_2') \circ_{W_3'} E_3') \cdots) \circ_{W_r} E_r' .$$

Conjecture. [McN, van Willigenburg; inspired by main result of BTvW]
Two skew diagrams E and E' satisfy $E \sim E'$ if and only if, for some r,

$$\begin{aligned} E &= ((\cdots(E_1 \circ_{W_2} E_2) \circ_{W_3} E_3) \cdots) \circ_{W_r} E_r \\ E' &= ((\cdots(E_1' \circ_{W_2'} E_2') \circ_{W_3'} E_3') \cdots) \circ_{W_r} E_r' , \text{ where} \end{aligned}$$

$\circ$ $E_i = W_i O_i W_i$ satsifies Hypotheses 1-4 for all i,
$\circ$ E_i' and W_i' denote either E_i and W_i, or E_i^* and W_i^*.

Main open problem

Theorem. [McN, van Willigenburg]
Skew diagrams $E_1, E_2, \ldots, E_r$
$E_i = W_i O_i W_i$ satisfies Hypotheses 1-5
E_i' and W_i' denote either E_i and W_i, or E_i^* and W_i^*.
Then

$$((\cdots(E_1 \circ_{W_2} E_2) \circ_{W_3} E_3) \cdots) \circ_{W_r} E_r \;\sim\; ((\cdots(E_1' \circ_{W_2'} E_2') \circ_{W_3'} E_3') \cdots) \circ_{W_r} E_r' \,.$$

Conjecture. [McN, van Willigenburg; inspired by main result of BTvW]
Two skew diagrams E and E' satisfy $E \sim E'$ if and only if, for some r,

$$\begin{aligned} E &= ((\cdots(E_1 \circ_{W_2} E_2) \circ_{W_3} E_3) \cdots) \circ_{W_r} E_r \\ E' &= ((\cdots(E_1' \circ_{W_2'} E_2') \circ_{W_3'} E_3') \cdots) \circ_{W_r} E_r' \,, \text{ where} \end{aligned}$$

- $\circ$ $E_i = W_i O_i W_i$ satsifies Hypotheses 1-4 for all i,
- $\circ$ E_i' and W_i' denote either E_i and W_i, or E_i^* and W_i^*.

True for $n \leq 20$.

Summary

- A definition of skew diagram composition. Encompasses the composition, amalgamated composition and staircase operations of RSvW.
- Theorem that generalizes all previous results.
  In particular, explains the 6 missing equivalences from HDL II.
- Conjecture for necessary and sufficient conditions for $E \sim E'$.