Edge Labellings of Partially Ordered Sets

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Thesis Defense

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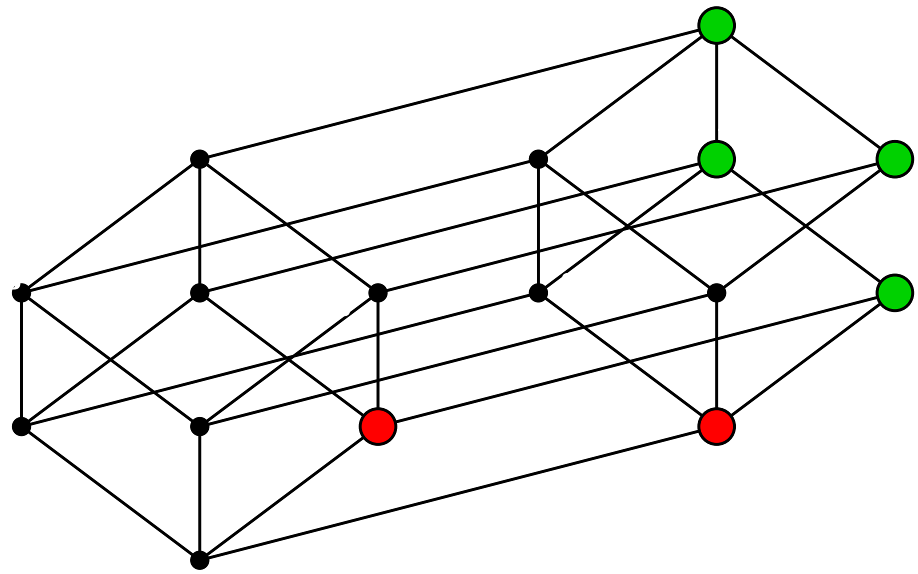
Slides available from

`http://www-math.mit.edu/~mcnamara/`

Outline:

1. Introduction
2. Supersolvable lattices and S_n EL-labellings
3. Actions on the maximal chains
4. Left modularity and extensions (joint with H. Thomas)

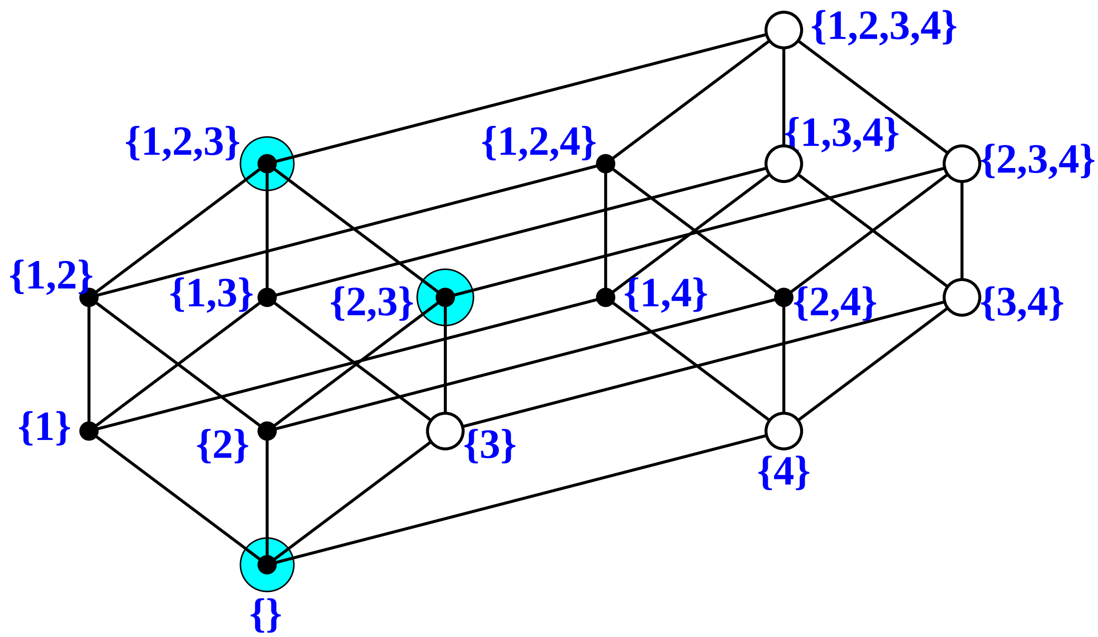


Definition A partially ordered set P is said to be a *lattice* if every two elements x and y of P have a least upper bound and a greatest lower bound.

We call the least upper bound the *join* of x and y and denote it by $x \vee y$.

We call the greatest lower bound the *meet* of x and y and denote it by $x \wedge y$.

(All our posets will be finite.)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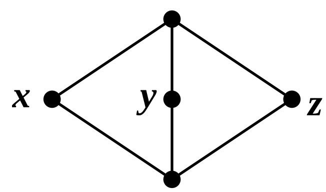
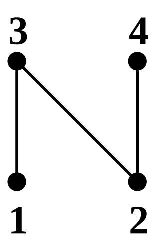
We say that a lattice L is *distributive* if

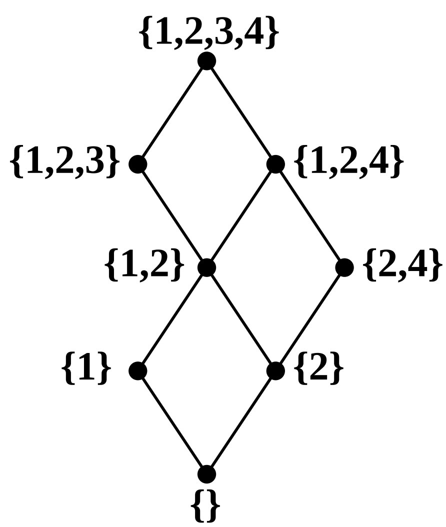
$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \qquad \text{and}$$
$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

for all elements x, y and z of L.

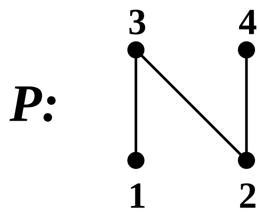


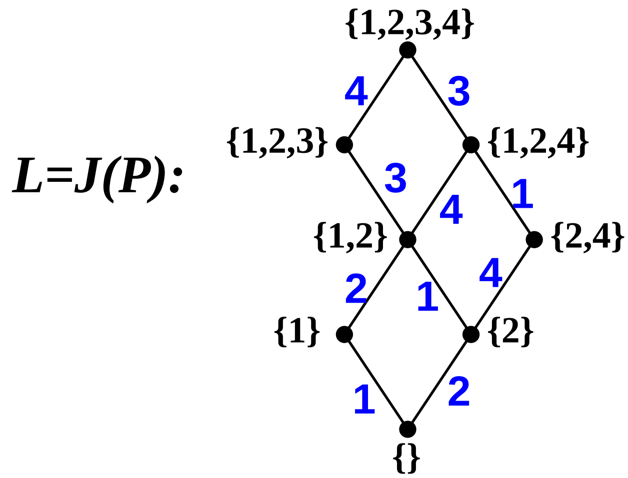
EXAMPLE An *order ideal* of a poset P is a subset I of P such that if $x \in I$ and $y \leq x$, then $y \in I$. The lattice of order ideals of a poset P is a distributive lattice.





Theorem *(FTFDL Birkhoff) A finite lattice L is distributive if and only if it is the lattice $J(P)$ of order ideals of some poset P.*

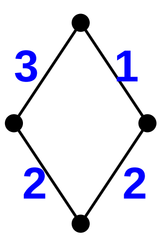




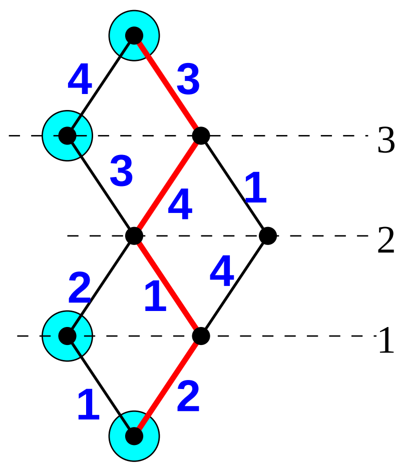
Definition An edge labelling of a poset P is said to be an *EL-labelling* if:

1. Every interval $[x, y]$ of P has exactly one maximal chain with increasing labels

2. The sequence of labels along this increasing maximal chain lexicographically precede the labels along any other maximal chain of $[x, y]$.



Who cares? P is a bounded graded poset of rank n. Let S be any subset of $\{1, 2, \ldots, n-1\}$.



- Flag f-vector $\alpha_P(S)$: number of chains in P with rank set S.

  If P has an EL-labelling: number of maximal chains of P with descent set contained in S.

- Flag h-vector $\beta_P(S)$:

$$\beta_P(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_P(T).$$

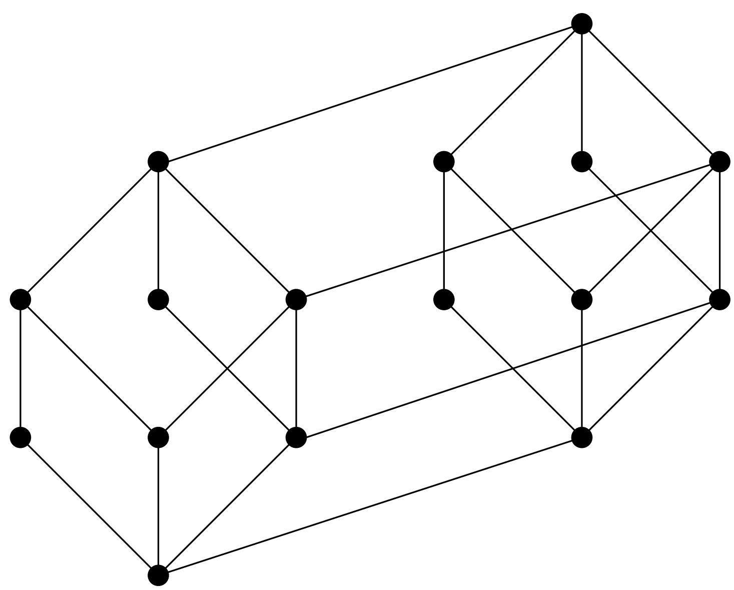
  If P has an EL-labelling: number of maximal chains of P with descent set S. So $\beta_P(S) \geq 0$.

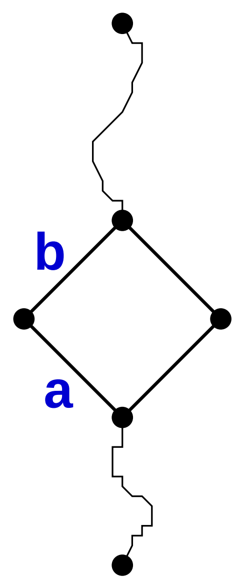
- Möbius function: $\mu(\hat{0}, \hat{1}) = (-1)^n \beta_P([n-1])$.

- EL-labelling $\Rightarrow$ Shellable $\Rightarrow$ Cohen-Macaulay

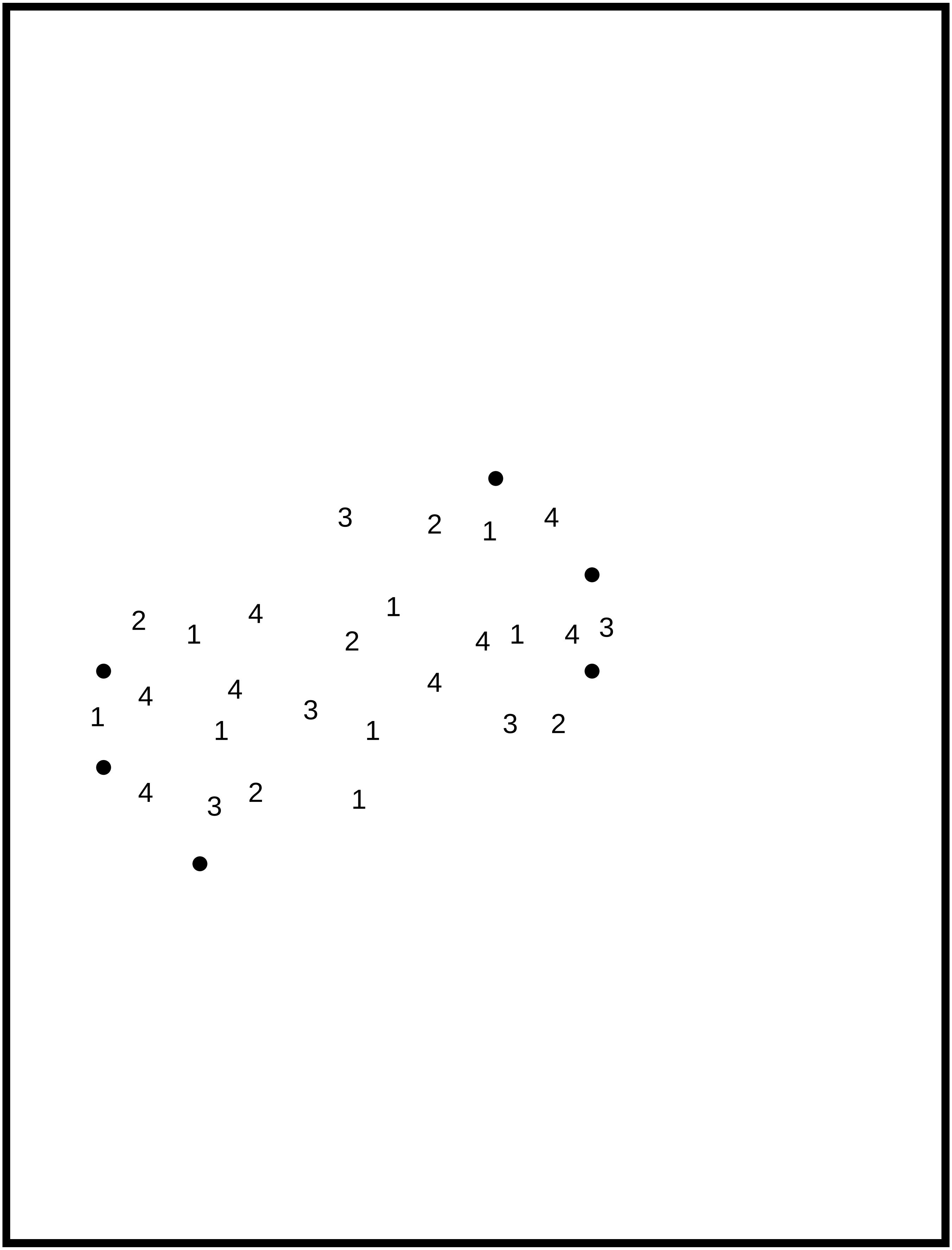
Definition An edge labelling of a poset P is said to be an *S_n EL-labelling* if:

1. Every interval $[x, y]$ of P has exactly one maximal chain with increasing labels
2. The labels along any maximal chain form a permutation of $\{1, 2, \ldots, n\}$.



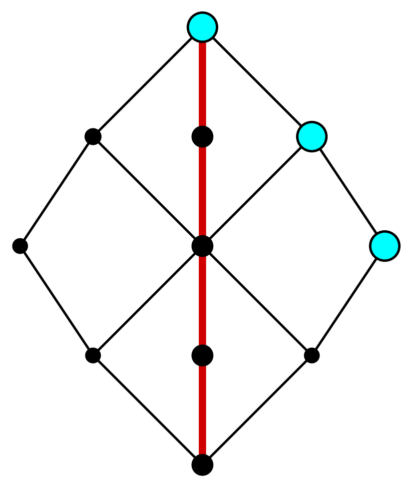


What other classes of posets have S_n EL-labellings?



Definition (R. Stanley, '72) A finite lattice L is said to be *supersolvable* if it contains a maximal chain $\mathfrak{m}_0$, called an *M-chain* of L which together with any other chain of L generates a distributive sublattice.

EXAMPLES



- Distributive lattices
- Modular lattices
- The lattice of partitions of $\{1, 2, \ldots, n\}$
- The lattice of non-crossing partitions of $\{1, 2, \ldots, n\}$
- The lattice of subgroups of a supersolvable group

QUESTION *"Are there any other lattices that have S_n EL-labellings?"*

THEOREM *A lattice is supersolvable if and only if it has an S_n EL-labelling.*

EXAMPLE Biagioli & Chapoton: Lattice of leaf labelled binary trees
`www.arxiv.org/math.CO/0304132`

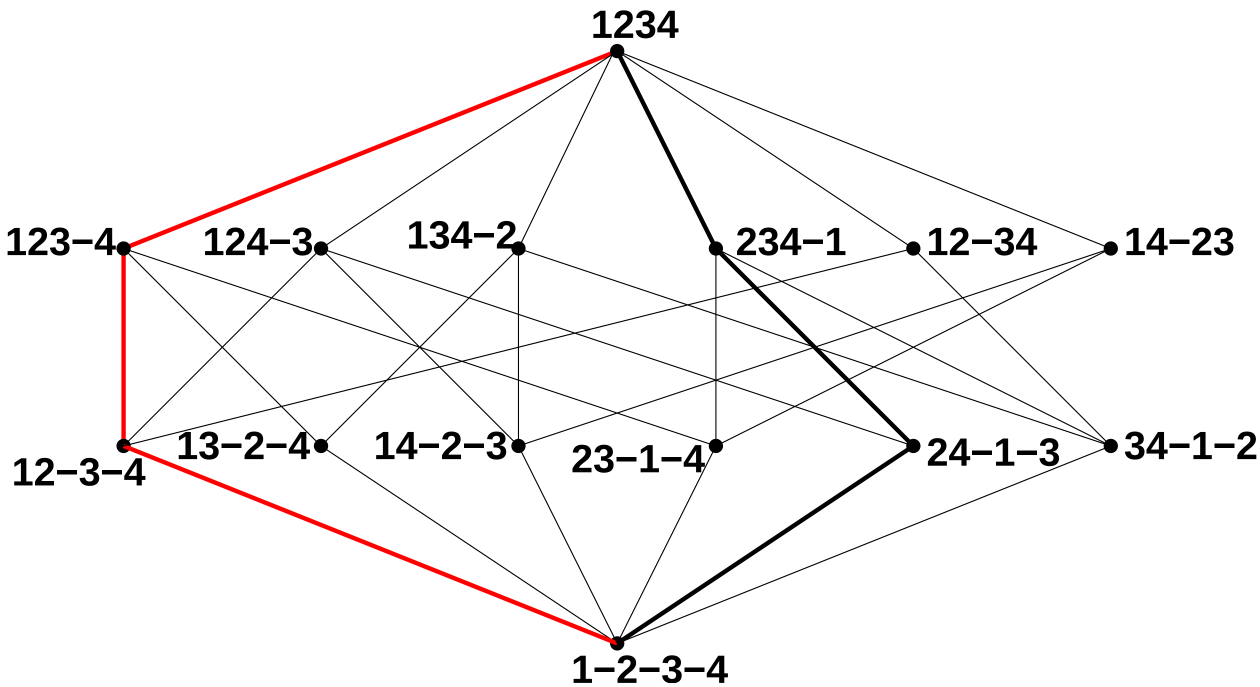
We want the chain $\mathfrak{m}_0$ with labels $1, 2, 3, \ldots, n$ to be an M-chain. Let $\mathfrak{m}$ be any other chain of L. (It suffices to consider only maximal chains.) The proof relies on the equivalence of the following 3 posets:

1. The sublattice $L_{\mathfrak{m}}$ of L generated by $\mathfrak{m}$ and $\mathfrak{m}_0$
2. Let $\omega_{\mathfrak{m}}$ be the permutation labelling $\mathfrak{m}$. Construct poset $P_{\omega_{\mathfrak{m}}}$ on $1, 2, \ldots, n$:

   $$i < j \text{ in } P_{\omega_{\mathfrak{m}}} \iff (i, j) \text{ isn't an inversion in } \omega_{\mathfrak{m}}$$

   for all $i < j$.
   Construct and label $J(P_{\omega_{\mathfrak{m}}})$ as before.
3. If $\mathfrak{m}$ doesn't have a descent at i:
   Let $U_i(\mathfrak{m}) = \mathfrak{m}$.
   If $\mathfrak{m}$ has a descent at i:
   Define $U_i(\mathfrak{m})$ to be the unique chain in L differing from $\mathfrak{m}$ only at rank i and having no descent at i.
   $Q_{\mathfrak{m}} :=$ "closure" of $\mathfrak{m}$ in L under the action of $U_1, U_2, \ldots, U_{n-1}$.

Example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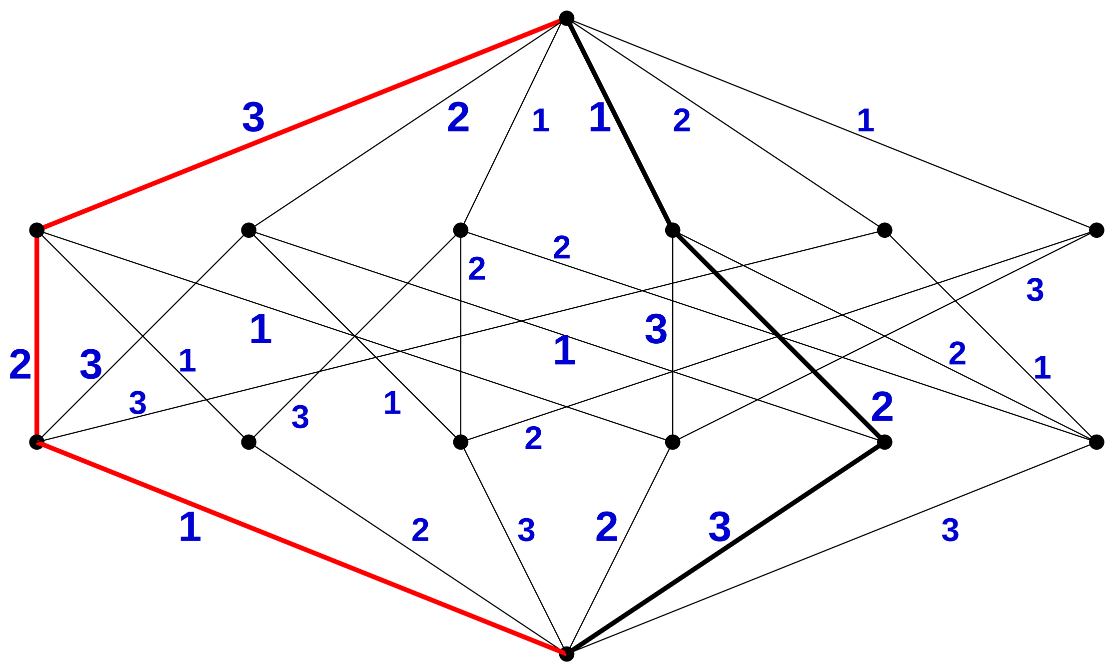


Leaving supersolvability behind...

Let P denote a bounded graded poset of rank n with an S_n EL-labelling.

The action of $U_1, U_2, \ldots, U_{n-1}$ has the following properties:

1. It is a local action: $U_i(\mathfrak{m})$ equals $\mathfrak{m}$ except possibly at rank i.
2. ${U_i}^2 = U_i$.
3. $U_i U_j = U_j U_i$ if $|i - j| \geq 2$.
4. $U_i U_{i+1} U_i = U_{i+1} U_i U_{i+1}$.

Compare with:

Definition The *Hecke algebra* of type A_{n-1} is the $\mathbb{C}(q)$-algebra generated by $T_1, T_2, \ldots, T_{n-1}$ with relations:

1. ${T_i}^2 = (q - 1)T_i + q$.
2. $T_i T_j = T_j T_i$ if $|i - j| \geq 2$.
3. $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$.

So we call our action a *local 0-Hecke algebra action*.

Duchamp, Hivert, Krob, Leclerc, Thibon.

One further desirable property:

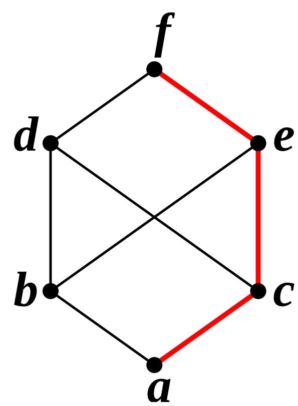
5. $\mathrm{ch}(\chi_P(x)) = \omega(F_P(x))$
   What the Hecke?
   Boils down to:
   Choose any subset S of $\{1, 2, \ldots, n-1\}$.
   Then the number $\alpha_P(S)$ of chains of P with rank set S equals the number of maximal chains fixed under U_i for all $i \notin S$.

An action on the maximal chains of a lattice having all of these properties is called a *good 0-Hecke algebra action*.

"Good": Simion and Stanley.

What other posets have good 0-Hecke actions?

EXAMPLE



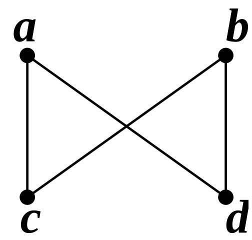
$\mathfrak{m}$	$U_1(\mathfrak{m})$	$U_2(\mathfrak{m})$	Fixed under
$\mathfrak{m}_1 : a < b < d < f$	$\mathfrak{m}_3$	$\mathfrak{m}_2$	$\emptyset$
$\mathfrak{m}_2 : a < b < e < f$	$\mathfrak{m}_4$	$\mathfrak{m}_2$	$\{2\}$
$\mathfrak{m}_3 : a < c < d < f$	$\mathfrak{m}_3$	$\mathfrak{m}_4$	$\{1\}$
$\mathfrak{m}_4 : a < c < e < f$	$\mathfrak{m}_4$	$\mathfrak{m}_4$	$\{1,2\}$

This gives a local 0-Hecke action. Also,

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{1,2\}$
$\alpha_P(S)$	1	2	2	4
# chains fixed under U_i for all $i \notin S$	1	2	2	4

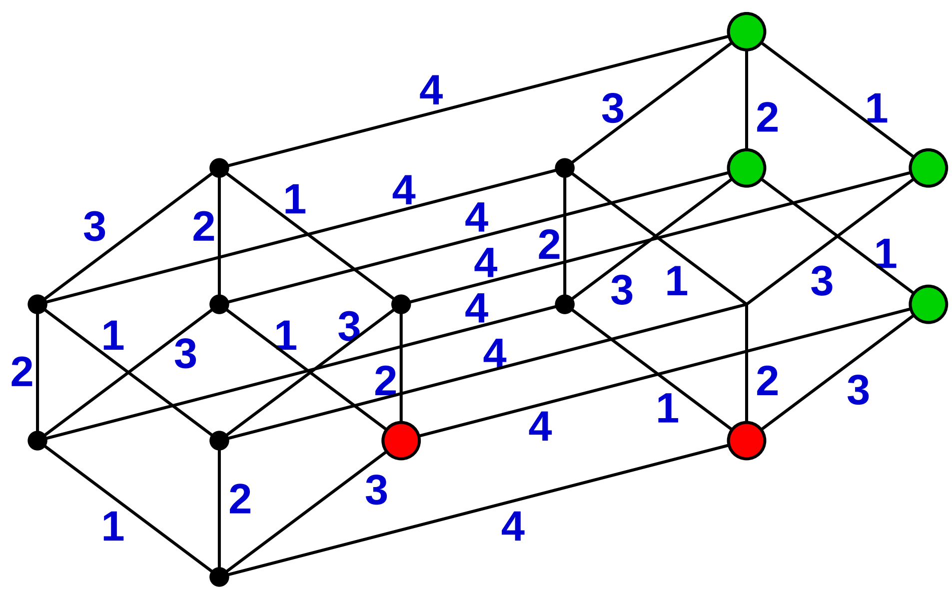
Therefore, this is a good 0-Hecke algebra action.

Definition A graded poset P is said to be *bowtie-free* if it does not contain distinct elements a, b, c, d connected by edges as shown:



EXAMPLE Lattices are bowtie-free.

EXAMPLE Non-lattices can be bowtie-free and also have an S_n EL-labelling:



THEOREM *Let P be a bounded graded bowtie-free poset of rank n. Then P has a good 0-Hecke algebra action if and only if P has an S_n EL-labelling.*

Idea of proof (Stanley):

1. Suppose P has a unique chain $\mathfrak{m}_0$ fixed under $U_1, U_2, \ldots, U_{n-1}$.
2. Given $\mathfrak{m}$ we can find $U_{i_1}, U_{i_2}, \ldots, U_{i_r}$ with r minimal such that $U_{i_1} U_{i_2} \cdots U_{i_r}(\mathfrak{m}) = \mathfrak{m}_0$.
3. Define $\omega_{\mathfrak{m}} = s_{i_1} s_{i_2} \cdots s_{i_r}$. Then $\omega_{\mathfrak{m}}$ is well-defined.
4. Label the edges of m from bottom to top by $\omega_{\mathfrak{m}}(1), \omega_{\mathfrak{m}}(2), \ldots, \omega_{\mathfrak{m}}(n)$. This gives an edge labelling of P and this edge labelling is an S_n EL-labelling.

Corollary *Let L be a lattice. TFAE:*

1. *L is supersolvable*
2. *L has an S_n EL-labelling*
3. *L has a good 0-Hecke algebra action*

Question *When does a **poset** have an S_n EL-labelling?*

Connections with modularity...

Suppose L is lattice with $y \leq z$. Always true:

$$(x \vee y) \wedge z \geq (x \wedge z) \vee y.$$

Definition An element x of a lattice L is said to be *left modular* if, for all $y \leq z$ in L, we have

$$(x \vee y) \wedge z = (x \wedge z) \vee y.$$

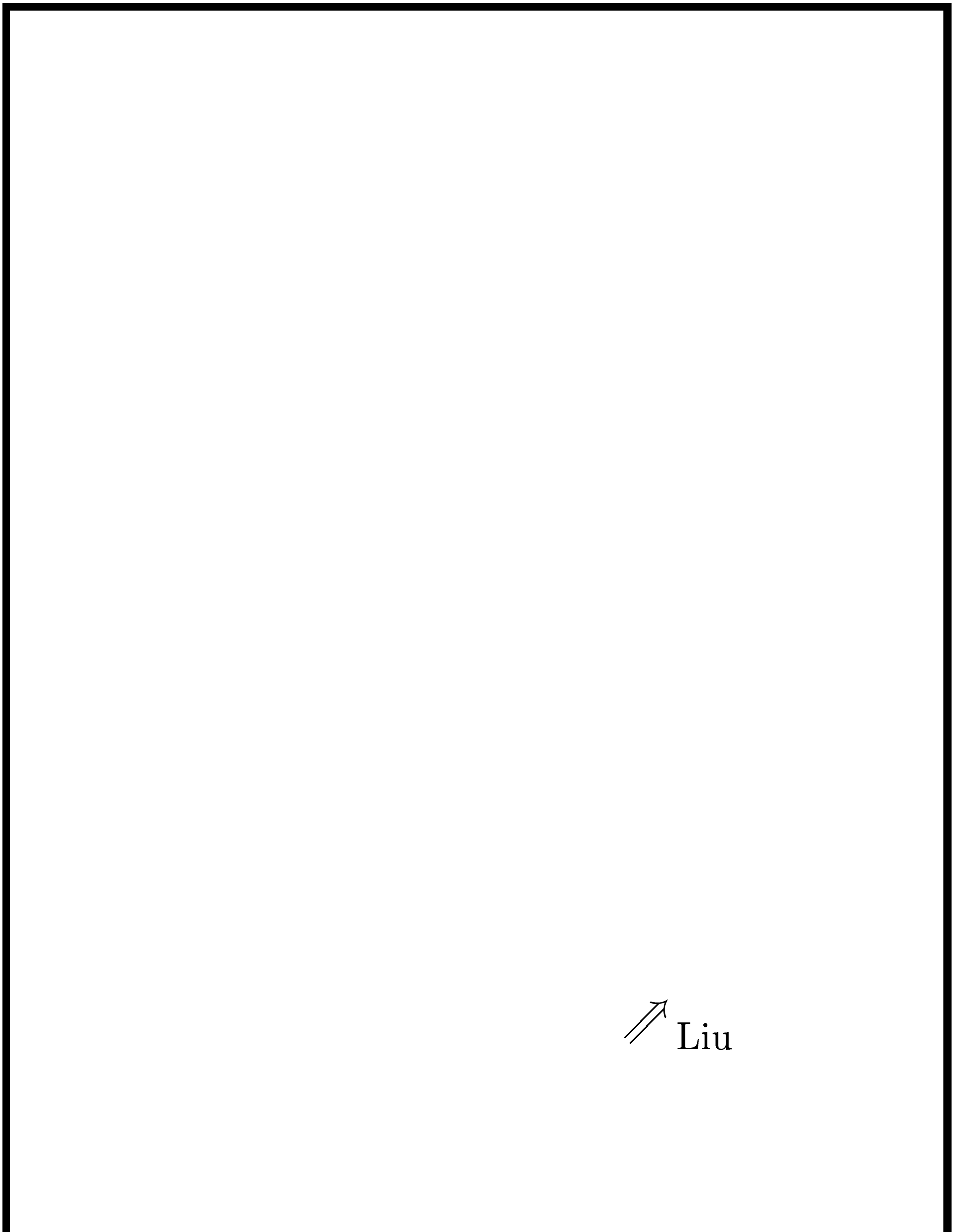
A chain of L is *left modular* if each of its elements is left modular.

Suppose L is a **graded** lattice.

L is supersolvable	$\Longleftrightarrow$	L has an S_n EL-labelling

L has a left modular maximal chain

Stanley $\Searrow$


Liu

THEOREM *Let L be graded lattice. TFAE:*

1. *L is supersolvable*
2. *L has an S_n EL-labelling*
3. *L has a good 0-Hecke algebra action*
4. *L has a left modular maximal chain*

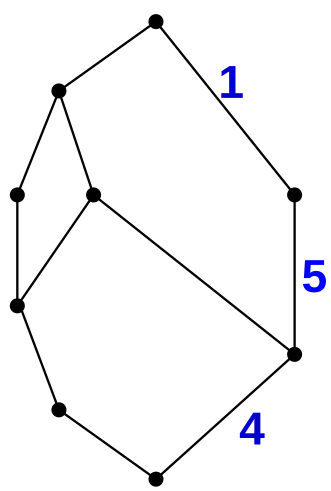
How can we extend this?

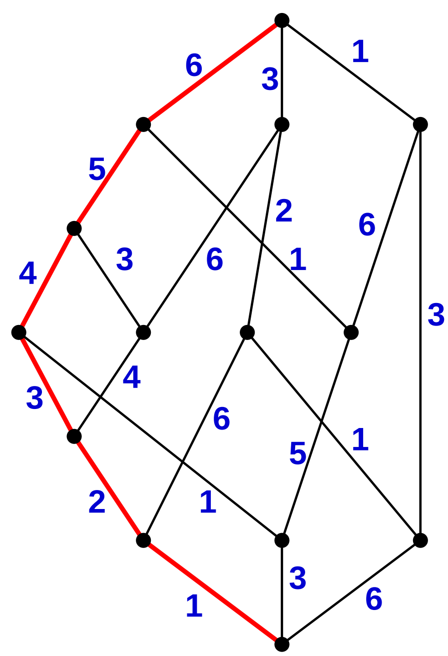
- 4: L need not be graded
- 2: L need not be a lattice

Definition Let P be a bounded poset. An EL-labelling γ of P is said to be *interpolating* if, for any $y \lessdot u \lessdot z$, either

(i) $\gamma(y, u) < \gamma(u, z)$ or

(ii) the increasing chain from y to z , say $y = w_0 \lessdot w_1 \lessdot \cdots \lessdot w_r = z$, has the properties that its labels are strictly increasing and that $\gamma(w_0, w_1) = \gamma(u, z)$ and $\gamma(w_{r-1}, w_r) = \gamma(y, u)$.

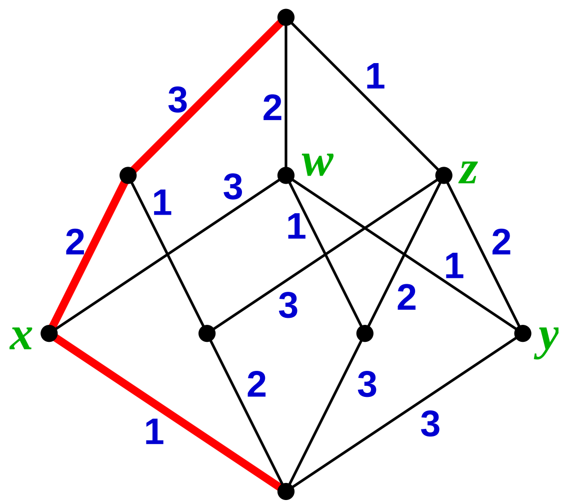




THEOREM *A lattice has an interpolating EL-labelling if and only if it has a left modular maximal chain.*

Generalizing to non-lattices:

P: a bounded poset with an S_n EL-labelling.
$\mathfrak{m}_0$: its increasing maximal chain.
Some "left modularity" property ?



When $x \in \mathfrak{m}_0$, $x \vee y$ and $x \wedge y$ are well-defined.

In a lattice: $(x \vee y) \wedge z \geq y$ whenever $z \geq y$.

When $x \in \mathfrak{m}_0$, $(x \vee y) \wedge_y z$ is well-defined for $y \leq z$. Similarly, $(x \wedge z) \vee^z y$ is well-defined.
We call x a *viable* element of P.
We call $\mathfrak{m}_0$ a *viable* maximal chain.

THEOREM *A bounded poset has an interpolating EL-labelling if and only if it has a viable left modular maximal chain.*

Finally, generalizing supersolvability:

Suppose P has a viable maximal chain $\mathfrak{m}_0$. So $(x \vee y) \wedge_y z$ and $(x \wedge z) \vee^z y$ are well-defined for $x \in \mathfrak{m}_0$ and $y \leq z$ in P.
Given any chain $\mathfrak{c}$ of P, we define $R_{\mathfrak{m}_0}(c)$ to be the smallest subposet of P satisfying:

(i) $\mathfrak{m}_0$ and $\mathfrak{c}$ are contained in $R_{\mathfrak{m}_0}(c)$,

(ii) If $y \leq z$ in P and y and z are in $R_{\mathfrak{m}_0}(c)$, then so are $(x \vee y) \wedge_y z$ and $(x \wedge z) \vee^z y$ for any x in $\mathfrak{m}_0$.

Definition We say that a finite bounded poset P is *supersolvable* with M-chain $\mathfrak{m}_0$ if $\mathfrak{m}_0$ is a viable maximal chain and $R_{\mathfrak{m}_0}(c)$ is a distributive lattice for any chain $\mathfrak{c}$ of P.

THEOREM *Let P be a bounded graded poset of rank n. TFAE:*

1. *P has an S_n EL-labelling*
2. *P has a viable left modular maximal chain*
3. *P is supersolvable*

	Graded	**Not nec. graded**
Lattice	1. Supersolvable 2. S_n EL-labelling 3. Good 0-Hecke action 4. Left mod. max. chain	1. ? 2. Interp. EL-labelling 3. ? 4. Left mod. max. chain
Not nec. Lattice	1. Supersolvable 2. S_n EL-labelling 3. Good 0-Hecke action (in bowtie-free case) 4. Viable left mod. m.c.	1. ? 2. Interp. EL-labelling 3. ? 4. Viable left mod. m.c.