Poset Edge-Labellings and Left Modularity

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Slides and papers available from

`http://www-math.mit.edu/~mcnamara/`

P: a partially ordered set (poset)
x, y: elements of P

If x and y have a least upper bound, then we call it the *join* of x and y and denote it by $x \vee y$.

If x and y have a greatest lower bound, then we call it the *meet* of x and y and denote it by $x \wedge y$.

A *lattice* is a poset in which every two elements have a meet and a join.

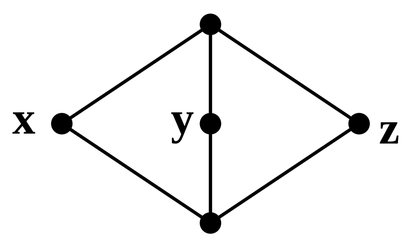
Definition We say that a lattice L is *distributive* if

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

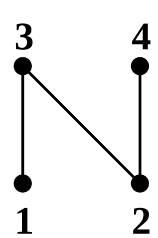
and

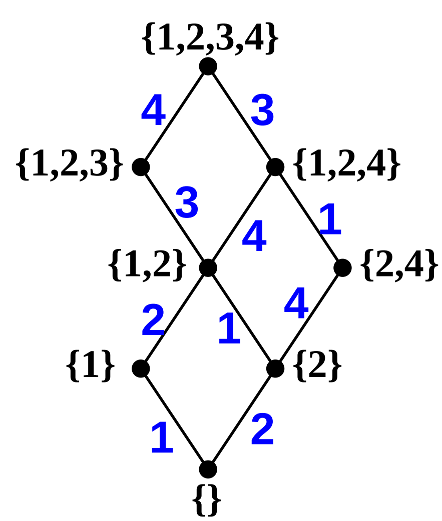
$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

for all elements x, y and z of L.



EXAMPLE The lattice of *order ideals* of a poset.



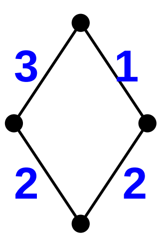


An edge-labelling of a poset P is said to be an *S_n EL-labelling* if:

1. Every interval $[x, y]$ of P has exactly one maximal chain with increasing labels
2. The labels along any maximal chain form a permutation of n.

Special case of *EL-labelling* (A. Björner):

2. The sequence of labels along this increasing maximal chain lexicographically precede the labels along any other maximal chain of $[x, y]$.



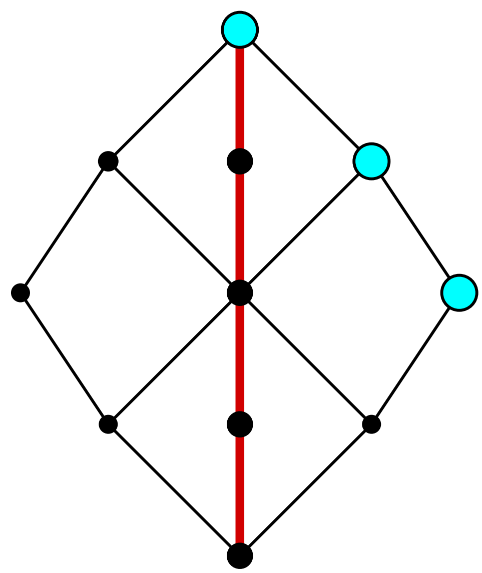
Who cares?

- EL-labelling ⇒ Shellable ⇒ Cohen-Macaulay
- Simple combinatorial interpretations of Möbius function, flag h-vector, etc.

What other classes of posets have S_n EL-labellings?

Definition(R. Stanley, 1972) A finite lattice L is said to be *supersolvable* if it contains a maximal chain $\mathfrak{m}$, called an *M-chain* of L, which together with any other chain of L generates a distributive sublattice.

EXAMPLES



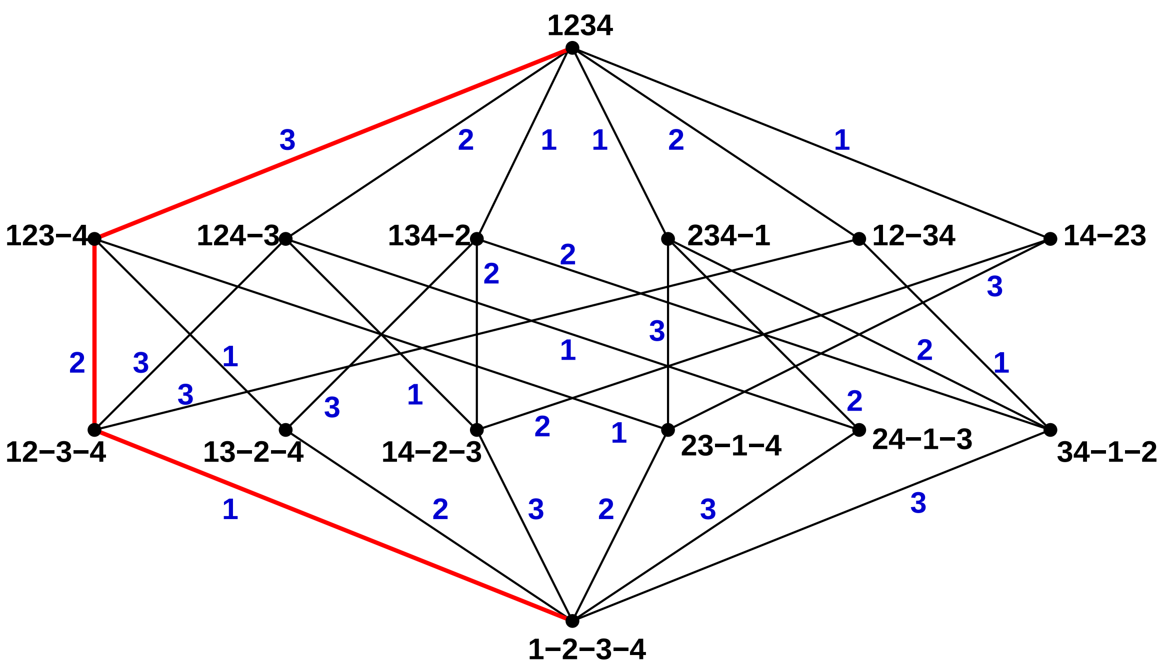
- Distributive lattices
- Modular lattices
- The lattice of partitions of $\{1, 2, \ldots, n\}$
- The lattice of subgroups of a supersolvable group

Question (Stanley) Are there any other lattices that have S_n EL-labellings?

Theorem (McN.) *A finite lattice has an S_n EL-labelling if and only if it is supersolvable.*

Examples

- Lattice of non-crossing partitions of $\{1, 2, \ldots, n\}$.



- Biagioli & Chapoton: Lattices of leaf labelled binary trees `www.arxiv.org/math.CO/0304132`

Connections with modularity…

Definition An element x of a lattice L is said to be *left-modular* if, for all $y \leq z$ in L, we have

$$(x \vee y) \wedge z = (x \wedge z) \vee y.$$

A chain of L is *left-modular* if each of its elements is left-modular.

Suppose L is a graded lattice.

L has an S_n EL-labelling	$\Longleftrightarrow$	L is supersolvable

L has a left-modular maximal chain

$\Swarrow$ Stanley

Liu $\Nwarrow$

Theorem *Let L be graded lattice. TFAE:*

1. *L is supersolvable*
2. *L has an S_n EL-labelling*
3. *L has a left-modular maximal chain*
4.

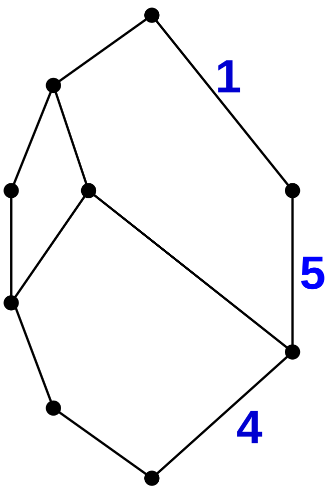
How can we extend this?

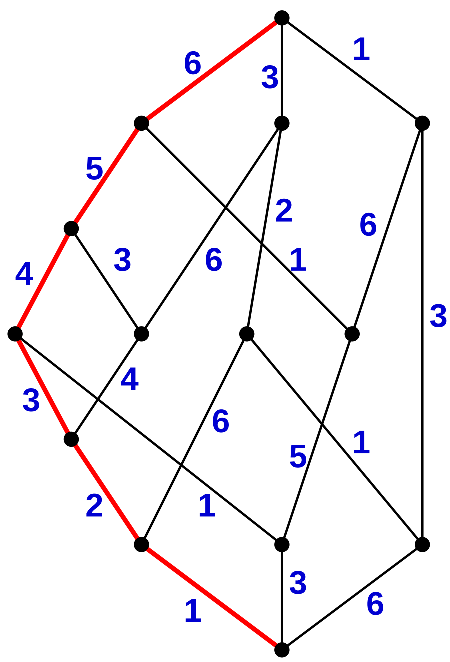
- 3: L need not be graded
- 2: L need not be a lattice

Definition Let P be a (bounded) poset. An EL-labelling γ of P is said to be *interpolating* if, for any $y \lessdot u \lessdot z$, either

(i) $\gamma(y,u) < \gamma(u,z)$ or

(ii) the increasing chain from y to z , say $y = w_0 \lessdot w_1 \lessdot \cdots \lessdot w_r = z$, has the properties that its labels are strictly increasing and that $\gamma(w_0, w_1) = \gamma(u,z)$ and $\gamma(w_{r-1}, w_r) = \gamma(y,u)$.





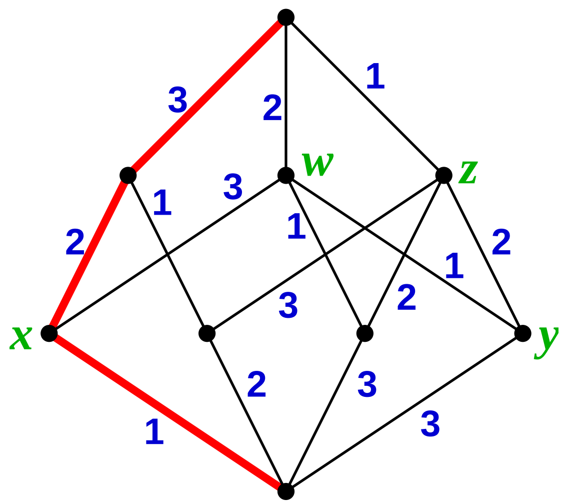
THEOREM (Thomas) *A lattice has an interpolating EL-labelling if and only if it has a left modular maximal chain.*

Generalizing to non-lattices:

P: a bounded poset with an S_n EL-labelling.

$\mathfrak{m}$: its increasing maximal chain.

Some "left modularity" property ?



When $x \in \mathfrak{m}$, $x \vee y$ and $x \wedge y$ are well-defined.

In a lattice: $(x \vee y) \wedge z \geq y$ whenever $z \geq y$.

When $x \in \mathfrak{m}$, $(x \vee y) \wedge_y z$ is well-defined for $y \leq z$.

Similarly, $(x \wedge z) \vee^z y$ is well-defined.

We call x a *viable* element of P.

We call $\mathfrak{m}$ a *viable* maximal chain.

THEOREM (McN.-Thomas) *A bounded poset has an interpolating EL-labelling if and only if it has a viable left modular maximal chain.*

Finally, generalizing supersolvability:

Suppose P has a viable maximal chain $\mathfrak{m}$. So $(x \vee y) \wedge_y z$ and $(x \wedge z) \vee^z y$ are well-defined for $x \in \mathfrak{m}$ and $y \leq z$ in P.
Given any chain $\mathfrak{c}$ of P, we define $R_{\mathfrak{m}}(\mathfrak{c})$ to be the smallest subposet of P satisfying:

(i) $\mathfrak{m}$ and $\mathfrak{c}$ are contained in $R_{\mathfrak{m}}(\mathfrak{c})$,

(ii) If $y \leq z$ in P and y and z are in $R_{\mathfrak{m}}(\mathfrak{c})$, then so are $(x \vee y) \wedge_y z$ and $(x \wedge z) \vee^z y$ for any x in $\mathfrak{m}$.

Definition We say that a finite bounded poset P is *supersolvable* with M-chain $\mathfrak{m}$ if $\mathfrak{m}$ is a viable maximal chain and $R_{\mathfrak{m}}(\mathfrak{c})$ is a distributive lattice for any chain $\mathfrak{c}$ of P.

THEOREM (McN.-Thomas) *Let P be a bounded graded poset of rank n. TFAE:*

1. *P has an S_n EL-labelling*
2. *P has a viable left modular maximal chain*
3. *P is supersolvable*

	Graded	**Not nec. graded**
Lattice	1. Supersolvable 2. S_n EL-labelling 3. Left mod. max. chain	1. ? 2. Interp. EL-labelling 3. Left mod. max. chain
Not nec. Lattice	1. “Supersolvable” 2. S_n EL-labelling 3. Viable left mod. m.c.	1. ? 2. Interp. EL-labelling 3. Viable left mod. m.c.