Infinite Log-Concavity: Developments and Conjectures

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Background and Motivation

Sequence: $(a_k) = a_0, a_1, \ldots, a_n$.
Let $a_i = 0$ for $i < 0$ and $i > n$.

(a_k) is **log-concave** if $a_k^2 \geq a_{k-1}a_{k+1}$ for all k.
e.g., rows of Pascal's triangle

Infinite log-concavity

- **$\mathcal{L}$-operator** on sequences defined by
  $\mathcal{L}(a_k) = (b_k)$ where $b_k = a_k^2 - a_{k-1}a_{k+1}$.
  e.g.,
  $$(a_k) = 0, 1, 4, 6, 4, 1, 0$$
  $$\mathcal{L}(a_k) = 0, 1, 10, 20, 10, 1, 0$$
  $$\mathcal{L}^2(a_k) = 0, 1, 80, 300, 80, 1, 0$$
- (a_k) is **i-fold log-concave** if $\mathcal{L}^i(a_k)$ is a nonnegative sequence.
  (So log-concavity = 1-fold log-concavity.)
- (a_k) is **infinitely log-concave** if $\mathcal{L}^i(a_k)$ is nonnegative for all $i \geq 0$.

Motivating conjecture [Boros and Moll]

The rows of Pascal's triangle are infinitely log-concave.

Kauers and Paule's result

The rows of Pascal's triangle are 5-fold log-concave.

Main Result

Main idea

Log-concavity is not preserved under $\mathcal{L}$.
e.g., $\mathcal{L}(4, 5, 4) = 16, 9, 16$.

So let $r \geq 1$. Say that (a_k) is **r-factor log-concave** if
$$a_k^2 \geq r\, a_{k-1}a_{k+1} .$$
Note that r-factor log-concavity implies log-concavity.

Theorem (r-factor log-concavity persists)

Let (a_k) be a nonnegative sequence and let
$$r = \frac{3+\sqrt{5}}{2} \approx 2.618 .$$
If (a_k) is r-factor log-concave, so is $\mathcal{L}(a_k)$.
Thus (a_k) is infinitely log-concave.

Corollary

The row $\binom{n}{k}_{k\geq 0}$ of Pascal's triangle is infinitely log-concave for all $n \leq 1450$.

Proof: Using a computer, repeatedly apply $\mathcal{L}$ to the row until it becomes r-factor log-concave.

1								
1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
1	8	28	56	70	56	28	8	1

Other Directions

Conjecture (columns)

The columns of Pascal's triangle are infinitely log-concave.

Conjecture (diagonals)

The diagonal $\binom{n+mu}{mv}_{m\geq 0}$ is infinitely log-concave if $u < v$.

The q-analogue $\left(\begin{bmatrix} n \\ k \end{bmatrix}\right)_{k\geq 0}$ of a row is not even 2-fold log-concave. (Now "nonnegative" means all coefficients are nonnegative.)

Conjecture (q-analogue of a column)

$\left(\begin{bmatrix} n \\ k \end{bmatrix}\right)_{n\geq k}$ is infinitely log-concave for all fixed k.

Theorem (symmetric functions)

$(h_k)_{k\geq 0}$ is 3-fold log-concave but is not 4-fold log-concave.

Conjecture [Fisk, M-S, Stanley] (real roots)

For $(a_k) = a_0, a_1, \ldots, a_n$ with $a_i \geq 0$
let $p[a_k] = a_0 + a_1x + \cdots a_nx^n$.

If $p[a_k]$ has all real roots, then so does $p[\mathcal{L}(a_k)]$.

Take-home idea

Infinite log-concavity is a natural concept deserving further study.