

Positivity Questions for Generalised Schur Functions

Le Séminaire du LaCIM

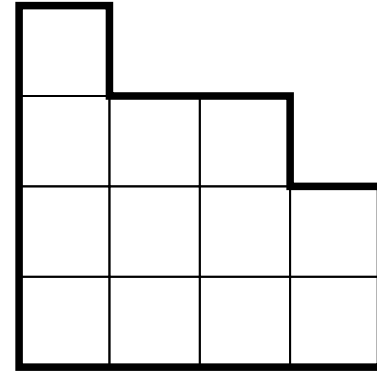
1 October 2004

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Slides and preprint available from
www.lacim.ugam.ca/~mcnamara

Schur functions

- Partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$.
- Example: $(4, 4, 3, 1)$

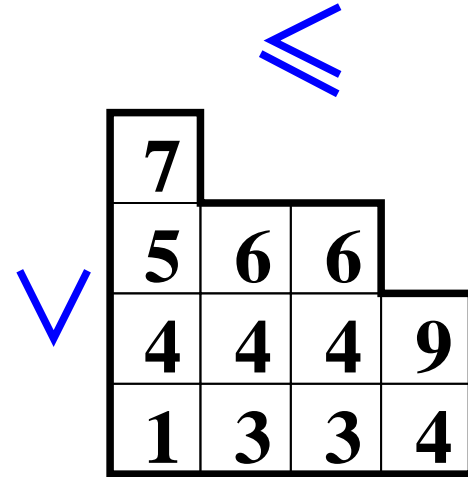


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- Partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$.

- Example: $(4, 4, 3, 1)$

- Semistandard Young tableau (SSYT)



Schur function s_λ in the variables

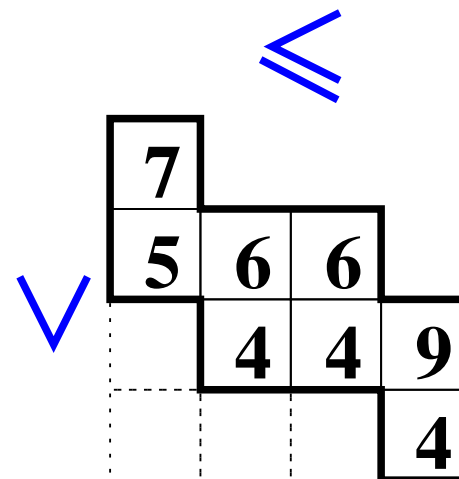
$\mathbf{x} = (x_1, x_2, \dots)$ defined by

$$s_\lambda(\mathbf{x}) = \sum_{\text{SSYT } T} \mathbf{x}^T = \sum_{\text{SSYT } T} x_1^{\#1\text{'s in } T} x_2^{\#2\text{'s in } T} \dots$$

$$s_{4431}(\mathbf{x}) = x_1^1 x_3^2 x_4^4 x_5 x_6^2 x_7 x_9 + \dots$$

Skew Schur functions

- Partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$.
- μ fits inside λ : form λ/μ .
- Example: $(4, 4, 3, 1)/(3, 1)$
- Semistandard Young tableau (SSYT)



Skew Schur function $s_{\lambda/\mu}$ in the variables $\mathbf{x} = (x_1, x_2, \dots)$ defined by

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Do we care?

- Schur functions are symmetric functions
- Schur functions s_λ form a basis for the symmetric functions.
- Arise in: representation theory of the symmetric group S_n .
- They are the characters of the irreducible representations of $GL(n, \mathbb{C})$.
- Correspond to Schubert classes in $H^*(Gr_{kn})$.

Do we care? For Schur!

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For skew Schur?

- Skew Schur functions are symmetric functions

$$s_{\lambda/\mu}(\mathbf{x}) = \sum_{\nu} c_{\lambda\mu}^{\nu} s_{\nu}(\mathbf{x}).$$

$c_{\lambda\mu}^{\nu}$: *Littlewood-Richardson coefficients*

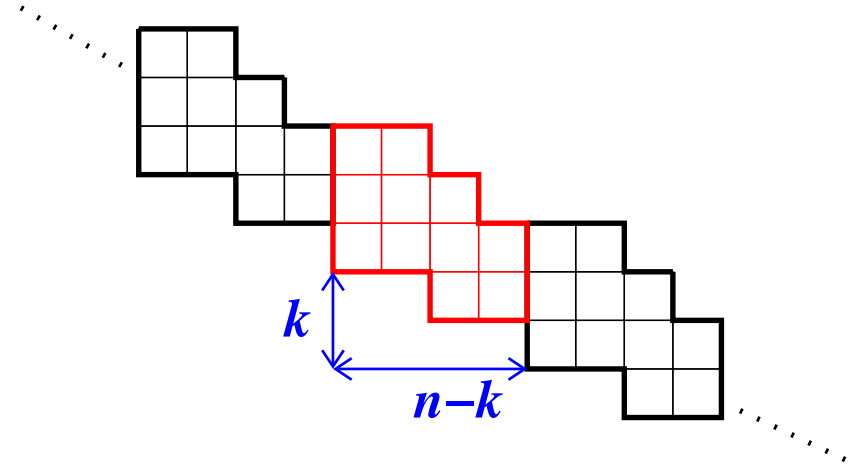
- Since $c_{\lambda\mu}^{\nu} \geq 0$, they are *Schur-positive*.

$$s_{4431/31} = s_{44} + 2s_{431} + s_{422} + s_{4211} + s_{332} + s_{3311}.$$

- Schur-positive symmetric functions are significant in the representation theory of S_n .

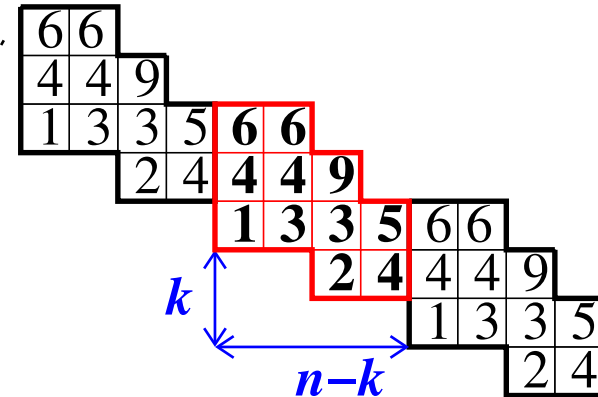
Cylindric skew Schur functions

- Infinite skew shape C
- Invariant under translation
- Identify (x, y) and $(x - n + k, y + k)$.



Cylindric skew Schur functions

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- Identify (x, y) and $(x - n + k, y + k)$.



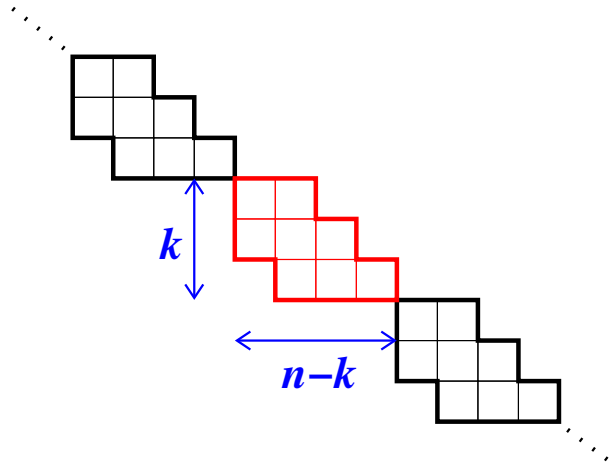
- Entries weakly increasing in each row
Strictly increasing up each column
- Alternatively: SSYT with relations between entries in first and last columns

$$s_C(\mathbf{x}) = \sum_T \mathbf{x}^T = \sum_T x_1^{\#1's \text{ in } T} x_2^{\#2's \text{ in } T} \dots$$

- s_C is a symmetric function

Cylindric skew Schur functions

EXAMPLE



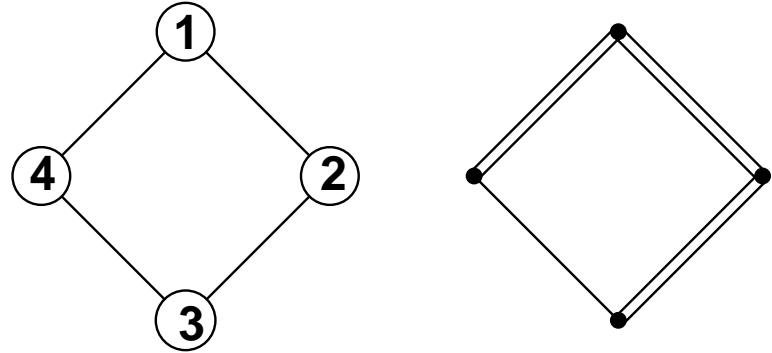
- Gessel, Krattenthaler: “*Cylindric Partitions*”
- Bertram, Ciocan-Fontanine, Fulton: “*Quantum Multiplication of Schur Polynomials*”
- Postnikov: “*Affine Approach to Quantum Schubert Calculus*” math.CO/0205165
- Stanley: “*Recent Developments in Algebraic Combinatorics*” math.CO/0211114

Motivation 1: P -partitions and an old conjecture of Stanley

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P : partially ordered set
(poset)

$\omega : P \rightarrow \{1, 2, \dots, |P|\}$
bijective labelling



DEFINITION (R. Stanley) Given a labelled poset (P, ω) , a (P, ω) -**partition** is a map $f : P \rightarrow \mathbb{P}$ with the following properties:

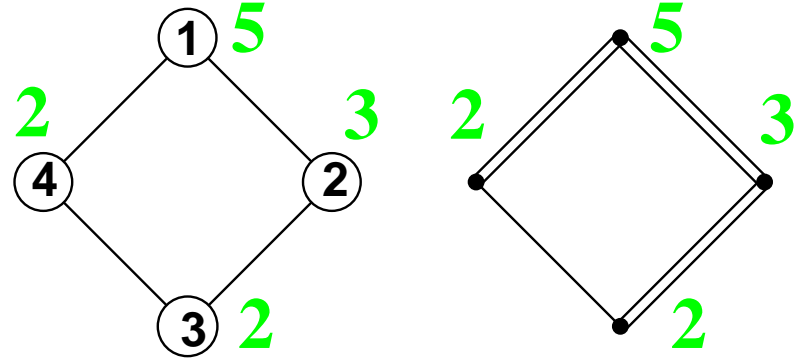
- f is *order-preserving*: If $x \leq y$ in P then $f(x) \leq f(y)$
- If $x \lessdot y$ in P and $\omega(x) > \omega(y)$ then $f(x) < f(y)$

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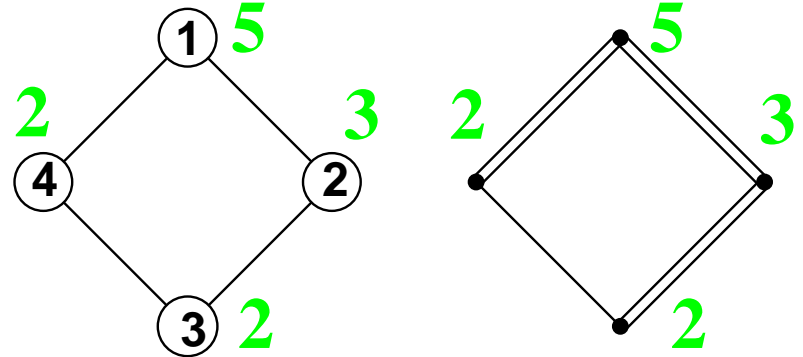
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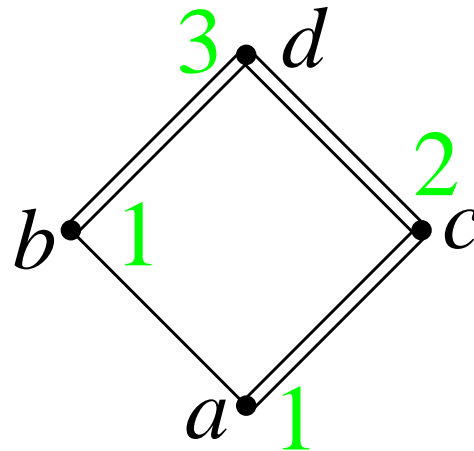
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- If $x < y$ in P and $\omega(x) > \omega(y)$ then $f(x) < f(y)$

$$K_{P, \omega}(\mathbf{x}) = \sum_f \mathbf{x}^T = \sum_f x_1^{\#f^{-1}(1)} x_2^{\#f^{-1}(2)} \dots$$

A non-symmetric example

$$K_{P,\omega}(\mathbf{x}) = \sum_f \mathbf{x}^T = \sum_f x_1^{\#f^{-1}(1)} x_2^{\#f^{-1}(2)} \dots$$

EXAMPLE

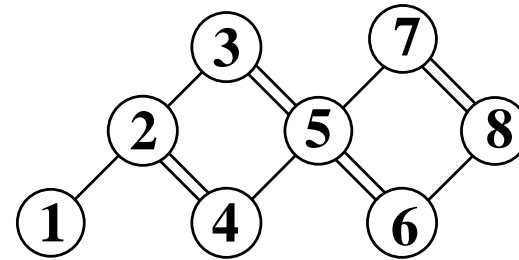
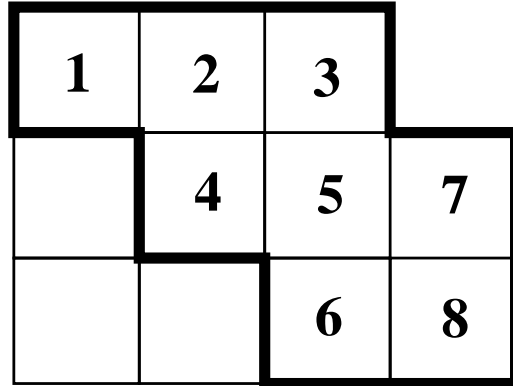


Coefficient of $x_1^2 x_2 x_3 = 1$

Coefficient of $x_1 x_2 x_3^2 = 0$

⇒ not symmetric

Schur labelled skew shape posets and Stanley's P -partitions Conjecture

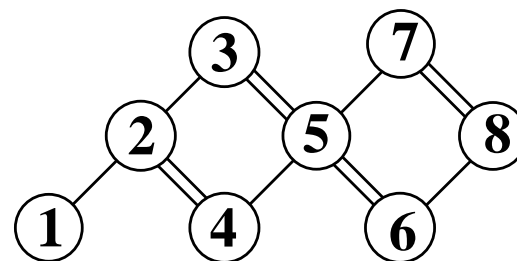
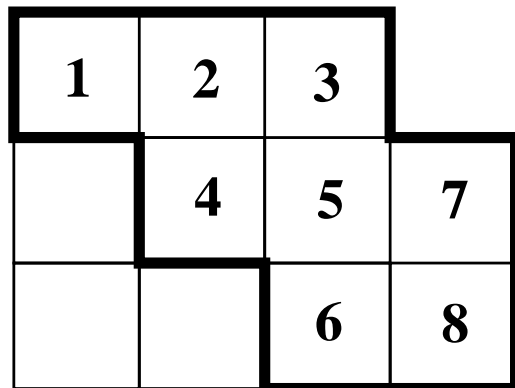


Bijection: SSYT of shape $\lambda/\mu \leftrightarrow (P, \omega)$ -partitions
 Furthermore,

$$K_{P, \omega}(\mathbf{x}) = s_{\lambda/\mu}(\mathbf{x}).$$

BIG QUESTION *What other labelled posets (P, ω) have symmetric $K_{P, \omega}(x)$?*

Schur labelled skew shape posets and Stanley's P -partitions Conjecture



Bijection: SSYT of shape $\lambda/\mu \leftrightarrow (P, \omega)$ -partitions
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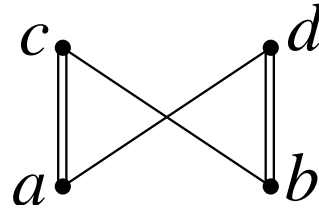
$$K_{P, \omega}(\mathbf{x}) = s_{\lambda/\mu}(\mathbf{x}).$$

BIG QUESTION What other labelled posets (P, ω) have symmetric $K_{P, \omega}(x)$?

CONJECTURE (Stanley, c.1971) $K_{P, \omega}(\mathbf{x})$ is symmetric if and only if (P, ω) is isomorphic to a Schur labelled skew shape poset.

Connection to cylindric skew Schur functions

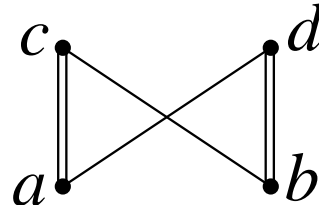
EXAMPLE



We can check that $K_{P,\omega}(x)$ is symmetric.
So does it obey Stanley's conjecture?

Connection to cylindric skew Schur functions

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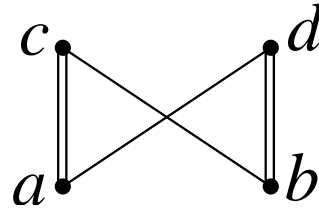


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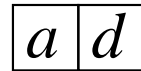
a

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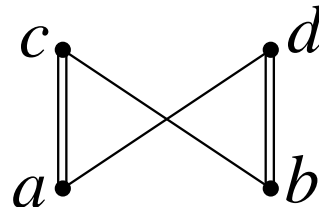


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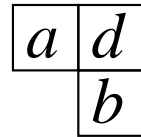


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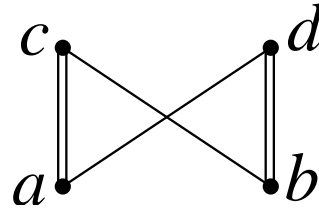


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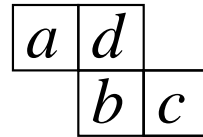


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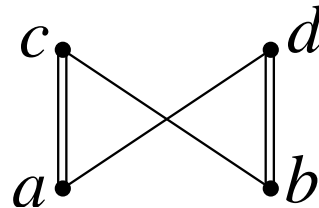


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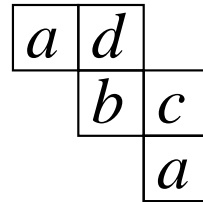


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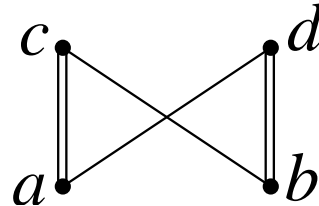


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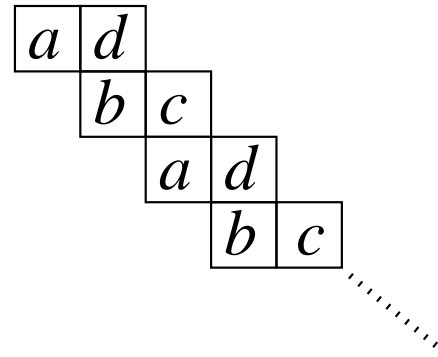


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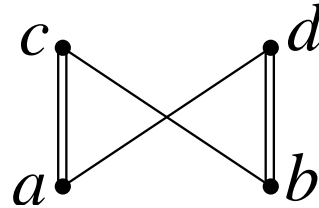


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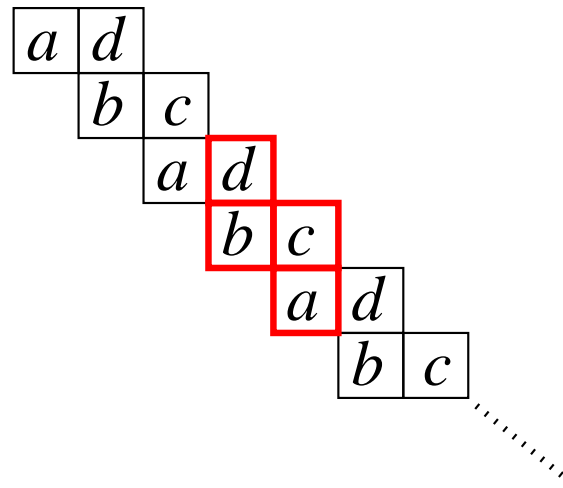


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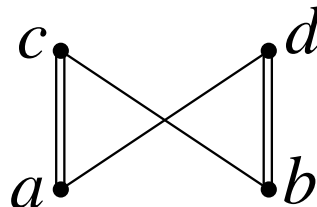


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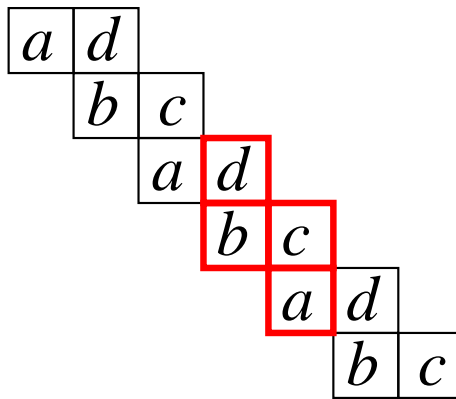


Connection to cylindric skew Schur functions

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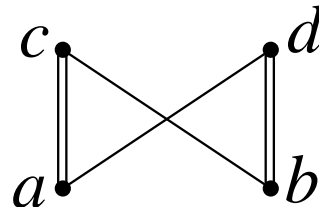
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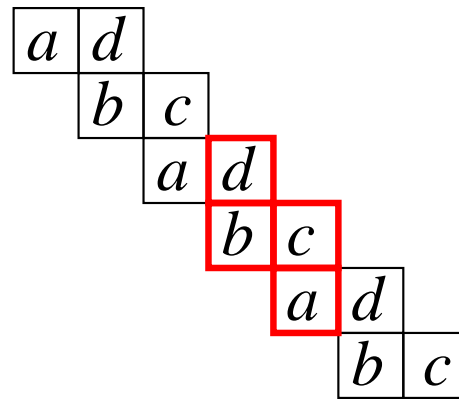
$\omega(a) > \omega(c) > \omega(b) > \omega(d) > \omega(a)$ Yikes!

Connection to cylindric skew Schur functions

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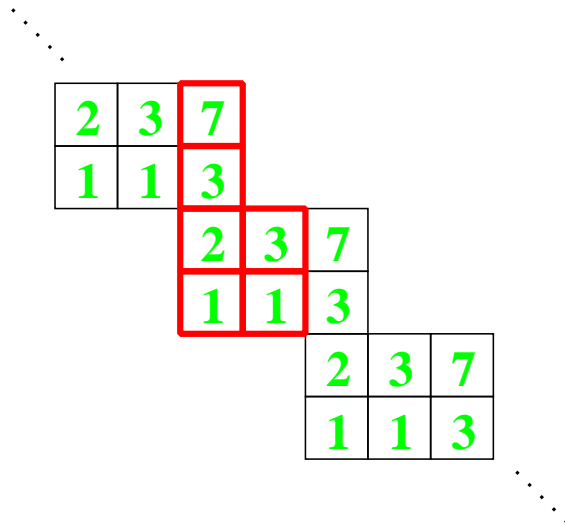
$\omega(a) > \omega(c) > \omega(b) > \omega(d) > \omega(a)$ Yikes! **Oriented Poset**

(P, O) -partitions

Labelled poset (P, ω)

$$K_{P, \omega}(\mathbf{x})$$

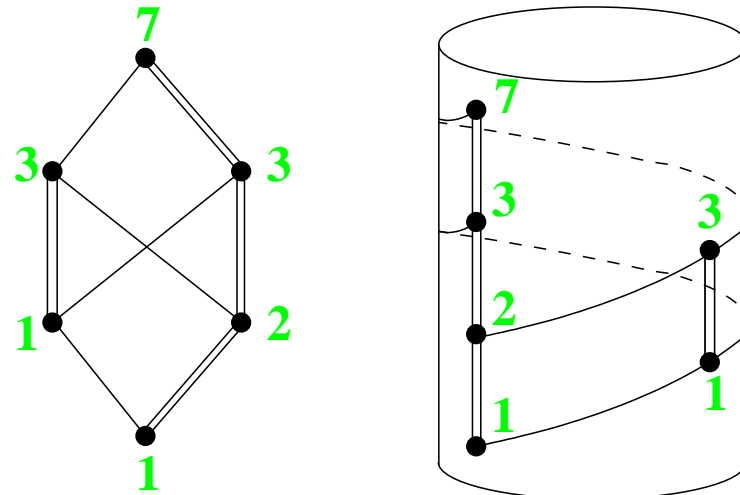
skew shape posets
skew Schur functions



Oriented poset (P, O)

$$K_{P, O}(\mathbf{x})$$

cylindric skew shape posets
cylindric skew Schur functions



Malvenuto's reformulation

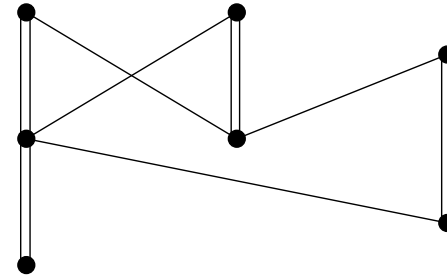
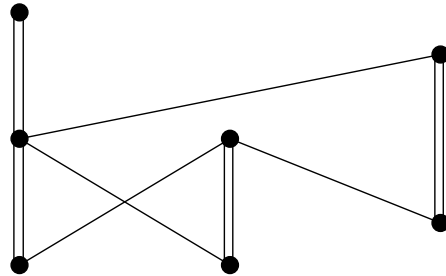
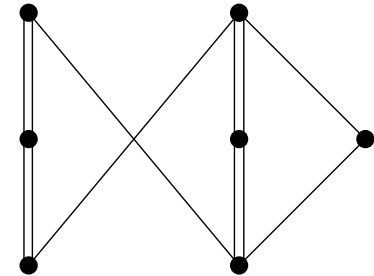
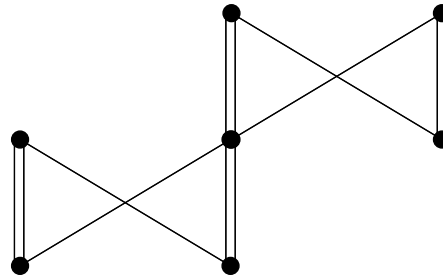
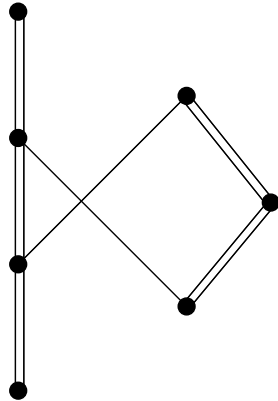
THEOREM (C. Malvenuto, c. 1995) *A labelled poset is a skew shape poset if and only if every connected component has no forbidden convex subposets*

CONJECTURE (*Stanley's conjecture restated*) *If (P, ω) has a forbidden convex subposet, then $K_{P, \omega}(\mathbf{x})$ is not symmetric.*

THEOREM (McN.) *An oriented poset is a cylindric skew shape poset if and only if every connected component has no forbidden convex subposets*

CONJECTURE (*Stanley's conjecture extended to oriented posets*) *$K_{P, O}(\mathbf{x})$ is symmetric if and only if every connected component of (P, O) is isomorphic to a cylindric skew shape poset.*

Extended version is false!



Motivation 2: Positivity of Gromov-Witten invariants

In $H^*(Gr_{kn})$,

$$\sigma_\lambda \sigma_\mu = \sum_{\nu \subseteq k \times (n-k)} c_{\lambda\mu}^\nu \sigma_\nu.$$

In $QH^*(Gr_{kn})$,

$$\sigma_\lambda * \sigma_\mu = \sum_{d \geq 0} \sum_{\substack{\nu \vdash |\lambda| + |\mu| - dn \\ \nu \subseteq k \times (n-k)}} q^d c_{\lambda\mu}^{\nu,d} \sigma_\nu.$$

$c_{\lambda\mu}^{\nu,d}$ = 3-point **Gromov-Witten invariants**

= $\#\{\text{rational curves of degree } d \text{ in } Gr_{kn} \text{ that meet fixed generic translates of the Schubert varieties } \Omega_\nu, \Omega_\lambda \text{ and } \Omega_\mu\}.$

Key point: $c_{\lambda\mu}^{\nu,d} \geq 0.$

“Fundamental Open Problem”:

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$c_{\lambda\mu}^{\nu,d}$ = 3-point **Gromov-Witten invariants**

= # {rational curves of degree d in Gr_{kn} that meet fixed generic translates of the Schubert varieties Ω_ν , Ω_λ and Ω_μ }.

Key point: $c_{\lambda\mu}^{\nu,d} \geq 0$.

“Fundamental Open Problem”: Find an algebraic or combinatorial proof of this fact.

What's cylindric got to do with it?

THEOREM (*Postnikov*)

$$s_{\lambda/d/\mu}(x_1, \dots, x_k) = \sum_{\nu \subseteq k \times (n-k)} C_{\lambda\mu}^{\nu,d} s_{\nu}(x_1, \dots, x_k).$$

Conclusion: Want to understand expansions of cylindric skew Schur functions into Schur functions.

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Conclusion: Want to understand expansions of cylindric skew Schur functions into Schur functions.

COROLLARY $s_{\lambda/d/\mu}(x_1, x_2, \dots, x_k)$ is Schur-positive.

Known: $s_{\lambda/d/\mu}(x_1, x_2, \dots)$ need **not** be Schur-positive.

Note: $s_{\lambda}(x_1, x_2, \dots, x_k) = 0 \Leftrightarrow \lambda$ has more than k rows.

Example: If $s_{\lambda/d/\mu} = s_{22} + s_{211} - s_{1111}$, then $s_{\lambda/d/\mu}(x_1, x_2, x_3) = s_{22} + s_{211}$ is Schur-positive.

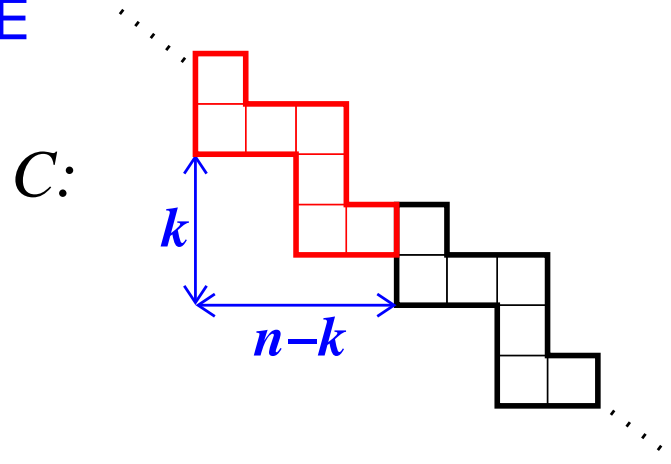
When is a cylindric skew Schur function Schur-positive?

THEOREM (McN.) *For any cylindric skew shape C ,
 $s_C(x_1, x_2, \dots)$ is Schur-positive $\Leftrightarrow C$ is a skew shape.*

Equivalently, if C is a non-trivial cylindric skew shape, then $s_C(x_1, x_2, \dots)$ is **not** Schur-positive.

Example: Cylindric ribbons

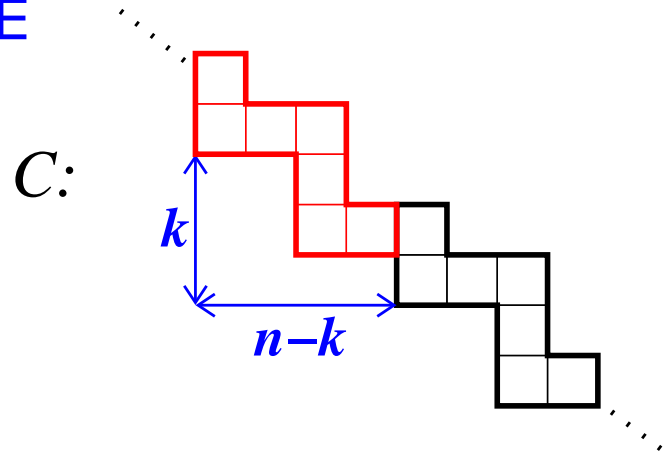
EXAMPLE



$$s_C(x_1, x_2, \dots) = \sum_{\nu \subseteq k \times (n-k)} c_\nu s_\nu + s_{n-k, 1^k} - s_{n-k-1, 1^{k+1}} \\ + s_{n-k-2, 1^{k+2}} - \dots + (-1)^{n-k} s_{1^n}.$$

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$$s_C(x_1, x_2, \dots) = \sum_{\nu \subseteq k \times (n-k)} c_\nu s_\nu + s_{n-k, 1^k} - s_{n-k-1, 1^{k+1}} \\ + s_{n-k-2, 1^{k+2}} - \dots + (-1)^{n-k} s_{1^n}.$$

Schur-positive with $k + 1$ variables

Not Schur-positive with $\geq k + 2$ variables

General cylindric skew shape: $\geq k + 2 + l$ variables

Shapes in Postnikov's theorem: $\geq 2k + 1$ variables

Tool: Cylindric skew Schur functions as signed sums of skew Schurs

Bertram, Ciocan-Fontanine, Fulton:

- 😊 Nice description in terms of ribbons
- 😞 Only for certain shapes, certain terms

Gessel, Krattenthaler:

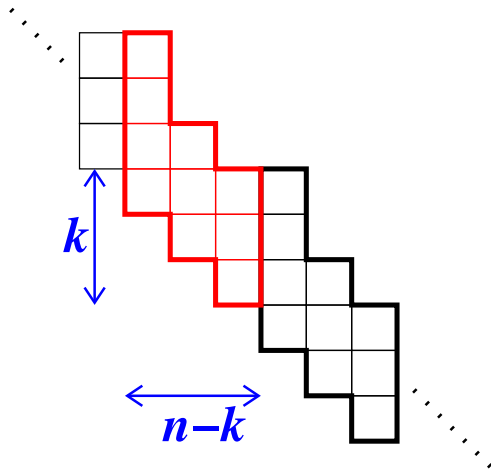
- 😊 Works for all cylindric skew shapes
- 😞 Not as nice a description

We can get the best of both worlds:

A technique for expanding a cylindric skew Schur function in terms of skew Schur functions that
Works for all cylindric skew shapes like G-K and
has a nice description like B-CF-F

Tool: Cylindric skew Schur functions as signed sums of skew Schurs

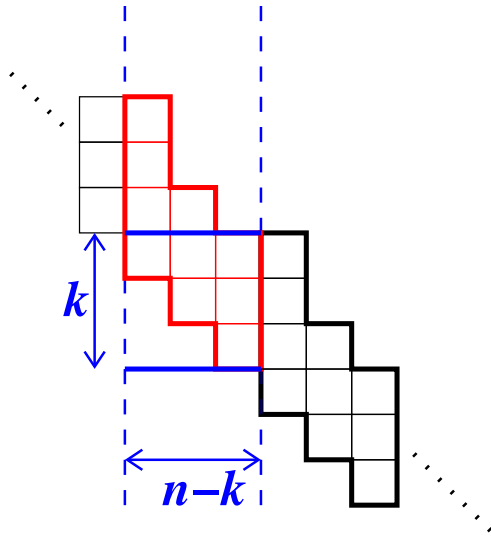
EXAMPLE



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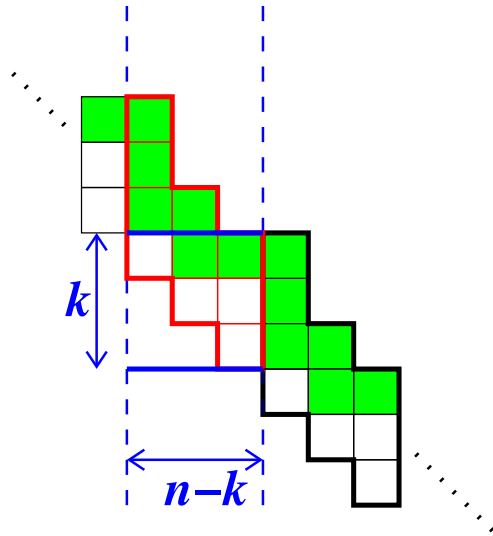
Tool: Cylindric skew Schur functions as signed sums of skew Schurs

EXAMPLE



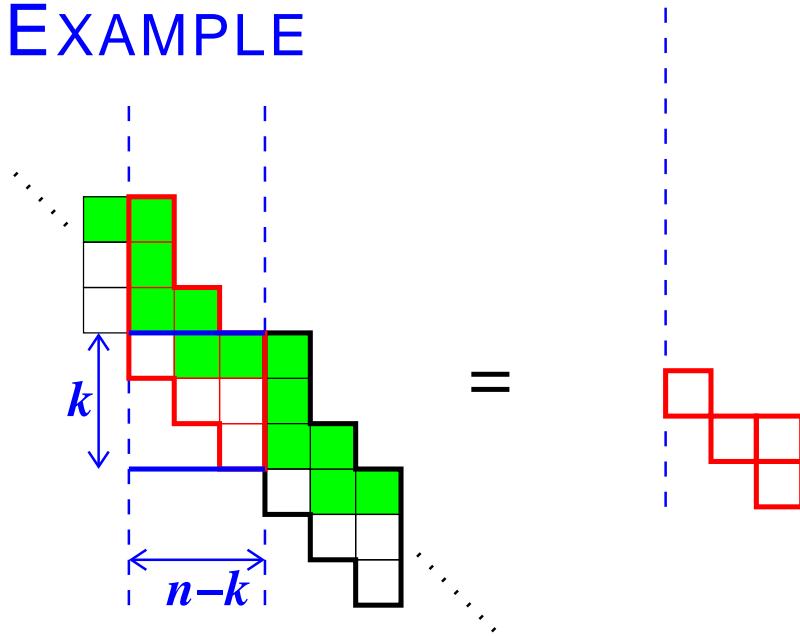
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EXAMPLE



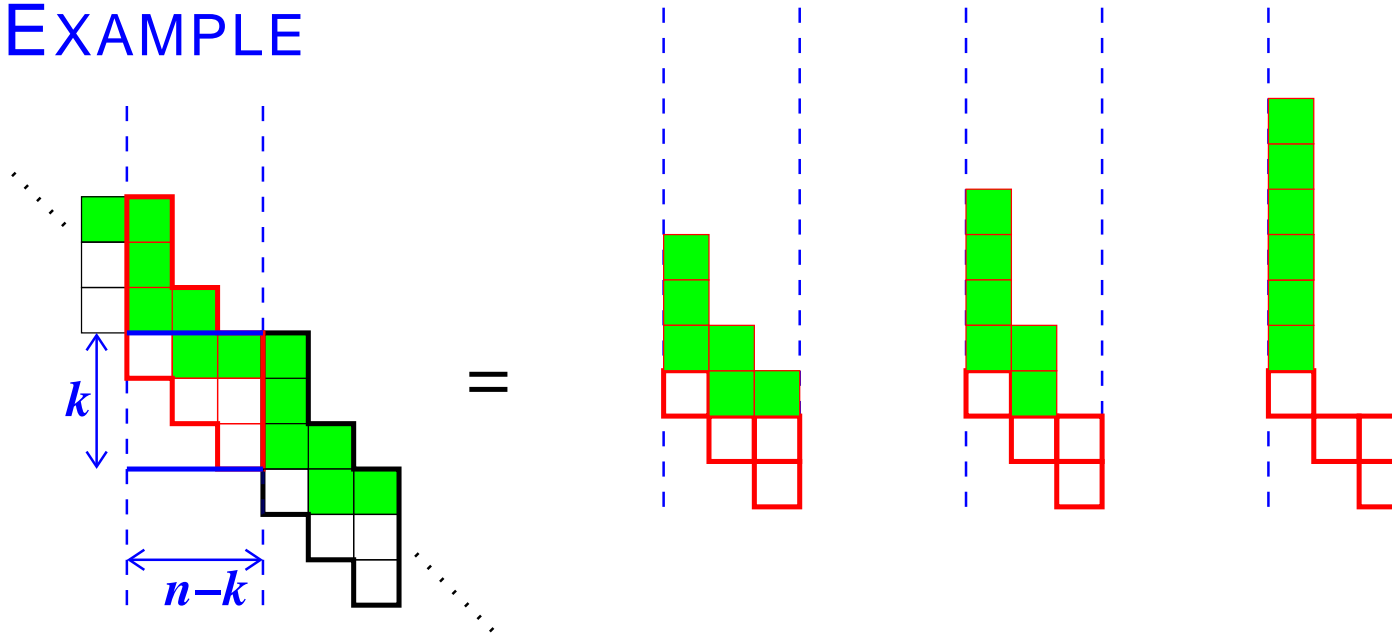
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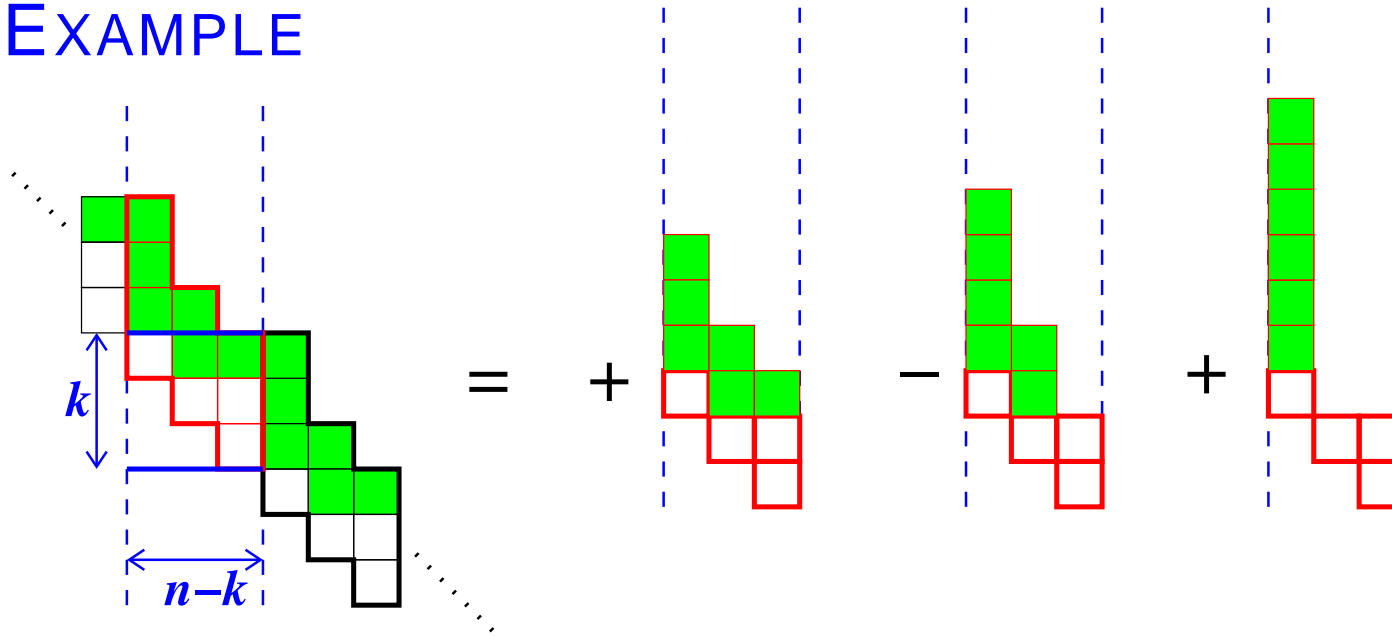
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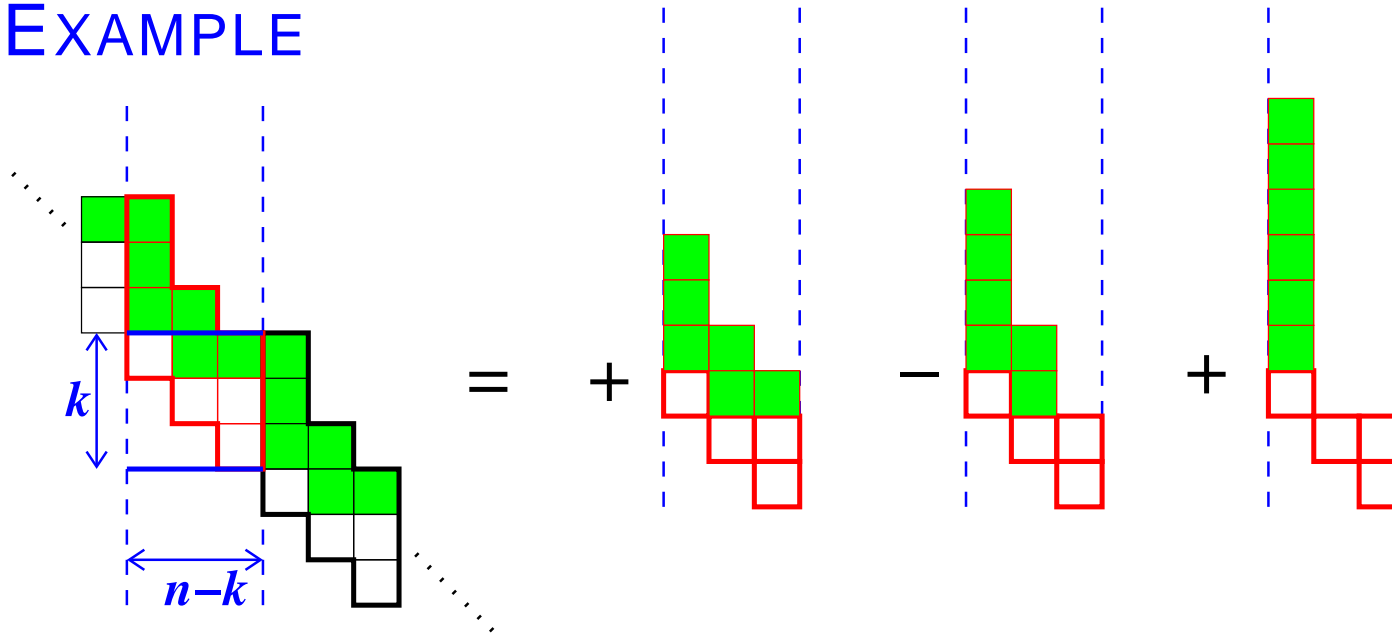
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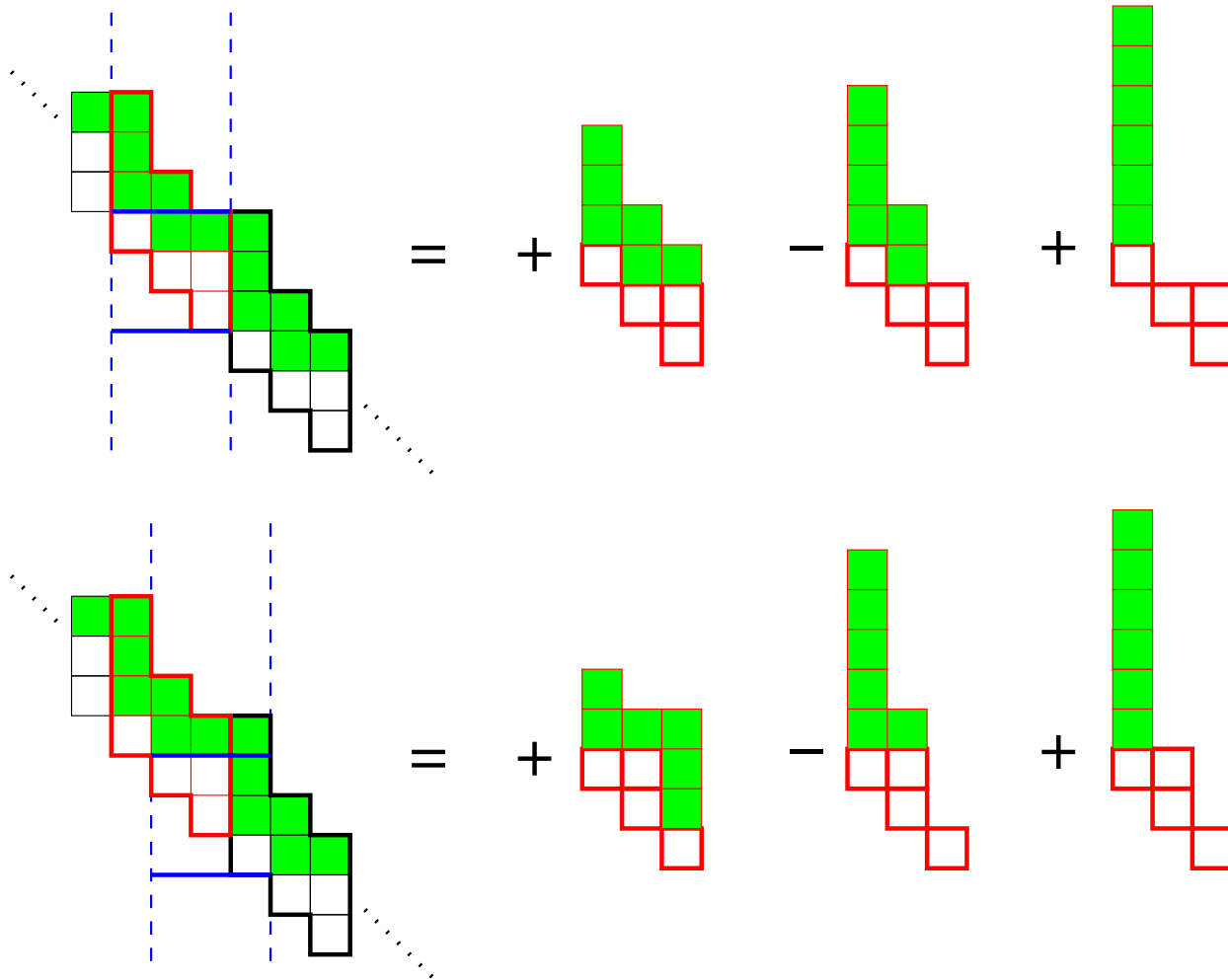
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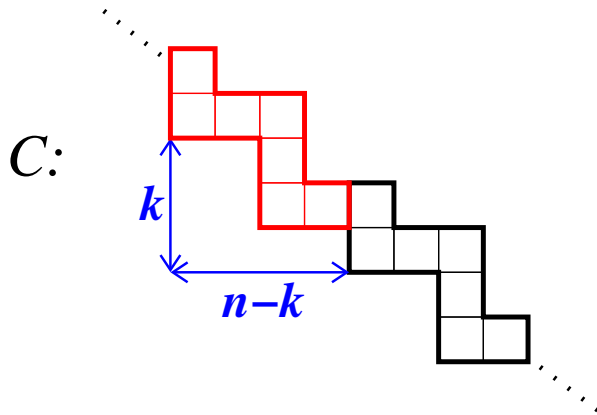


$$\begin{aligned}
 s_C &= s_{333211/21} - s_{3322111/21} + s_{331111111/21} \\
 &= s_{3331} + s_{3322} + s_{33211} + s_{322111} + s_{31111111} \\
 &\quad - s_{222211} - s_{2221111} + s_{22111111} + s_{211111111}.
 \end{aligned}$$

Consequence: Lots of skew Schur function identities

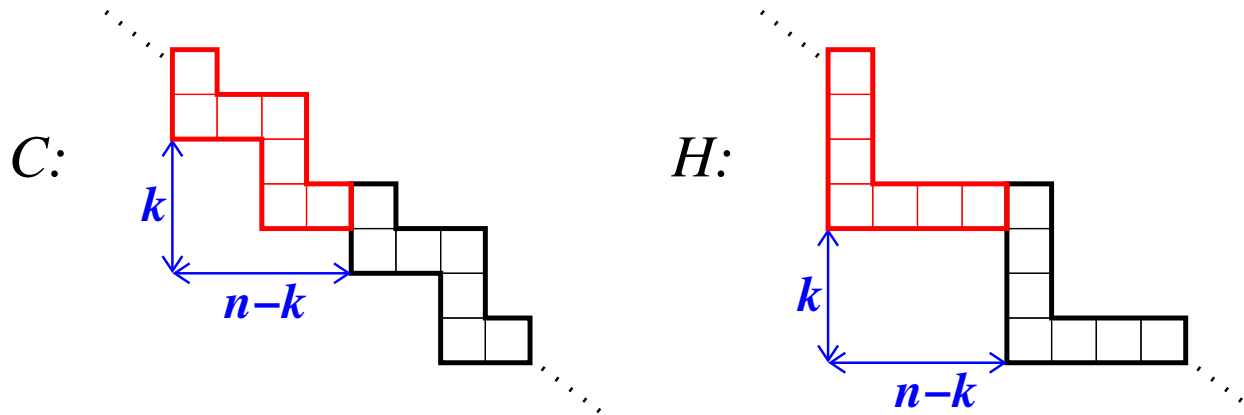


Example: Cylindric ribbons



$$\begin{aligned}
 s_C(x_1, x_2, \dots) &= \sum_{\nu \subseteq k \times (n-k)} c_\nu s_\nu + s_{n-k, 1^k} - s_{n-k-1, 1^{k+1}} \\
 &\quad + s_{n-k-2, 1^{k+2}} - \dots + (-1)^{n-k} s_{1^n}.
 \end{aligned}$$

Example: Cylindric ribbons



$$s_C(x_1, x_2, \dots) = \sum_{\nu \subseteq k \times (n-k)} c_\nu s_\nu + s_{n-k, 1^k} - s_{n-k-1, 1^{k+1}} \\ + s_{n-k-2, 1^{k+2}} - \dots + (-1)^{n-k} s_{1^n}.$$

However,
$$s_C(x_1, x_2, \dots) = \sum_{\nu \subseteq k \times (n-k)} c_\nu s_\nu + s_H.$$

s_C : cylindric skew Schur function

s_H : cylindric Schur function

We say that s_C is **cylindric Schur-positive**.

A Conjecture

CONJECTURE *For any cylindric skew shape C , s_C is cylindric Schur-positive.*

THEOREM (McN.) *The cylindric Schur functions corresponding to a given translation $(-n + k, +k)$ are linearly independent.*

THEOREM (McN.) *If s_C can be written as a linear combination of cylindric Schur functions with the same translation as C , then s_C is cylindric Schur-positive.*