Questions of Schur-Positivity

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joint project with

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Slides available from

`http://www.lacim.uqam.ca/~mcnamara/`

$\mathbf{x} = (x_1, x_2, \ldots)$

$f(\mathbf{x})$ is a symmetric function if f has finite degree and

$$f(x_1, x_2, \ldots) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots)$$

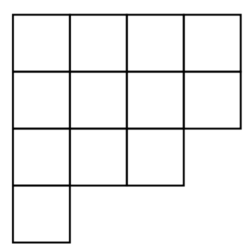
for all permutations σ of the positive integers.

EXAMPLE $f(\mathbf{x}) = \sum_{i \neq j} x_i^2 x_j$ is symmetric but $\sum_{i<j} x_i^2 x_j$ is not.

Λ: the ring of symmetric functions

A partition λ of a non-negative integer n is a sequence $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ of non-negative integers such that $\lambda_1 \geq \cdots \geq \lambda_k$ and $\sum_i \lambda_i = n$.

EXAMPLE $\lambda = (4, 4, 3, 1)$ is represented by its Young diagram as:



Bases for Λ:

- Monomial Symmetric Functions:

$$m_\lambda = \sum_\alpha x_1^{\alpha_1} x_2^{\alpha_2} \ldots$$

where the sum ranges over all permutations $\alpha = (\alpha_1, \alpha_2, \ldots)$ of the vector $\lambda = (\lambda_1, \lambda_2, \ldots)$.

Example

$$m_{(1)} \equiv m_1 = x_1 + x_2 + \cdots .$$

Example

$$m_{(2,1)} \equiv m_{21} = \sum_{i<j} x_i^2 x_j + \sum_{i<j} x_i x_j^2 = \sum_{i \neq j} x_i^2 x_j$$

- Elementary Symmetric Functions:

$$e_n = \sum_{i_1 < \cdots < i_n} x_{i_1} \ldots x_{i_n}, \qquad e_\lambda = e_{\lambda_1} e_{\lambda_2} \ldots .$$

- Complete Homogeneous Symmetric Functions:

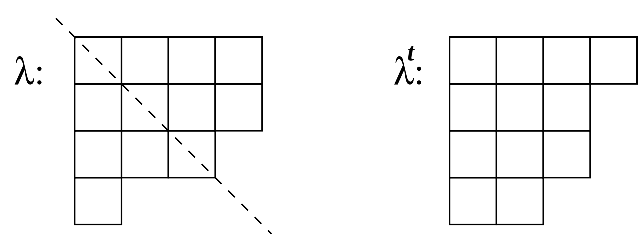
$$h_n = \sum_{i_1 \leq \cdots \leq i_n} x_{i_1} \ldots x_{i_n}, \qquad h_\lambda = h_{\lambda_1} h_{\lambda_2} \ldots .$$

- Power Sum Symmetric Functions:

$$p_n = \sum_i x_i^n, \qquad p_\lambda = p_{\lambda_1} p_{\lambda_2} \ldots .$$

NOTE $b_\lambda b_\mu = b_{\lambda_1} b_{\lambda_2} \ldots b_{\mu_1} b_{\mu_2} \ldots = b_{\lambda \sqcup \mu}$ if $b = e, h$ or p.

- Schur functions s_λ
  - Cauchy, 1815
  - Representation theory: symmetric group, general linear group, special linear group
  - Algebraic geometry: cohomology ring of the Grassmannian
  - Linear algebra: eigenvalues of Hermitian matrices
  - $$\langle s_\lambda, s_\mu \rangle = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{otherwise} \end{cases}$$
  - $\omega(s_\lambda) = s_{\lambda^t}$



  - Lots of interesting problems!

Semistandard Young Tableaux of shape λ, SSYT(λ):

$\leqslant$

***T*:** $\wedge$

2	2	5	5
3	4	6	7
7	7	9	
9			

Denote

$$\mathbf{x}^T = x_1^{\#1\text{'s in } T} x_2^{\#2\text{'s in } T} \cdots .$$

Then

$$s_\lambda(\mathbf{x}) = \sum_{T \in \mathrm{SSYT}(\lambda)} \mathbf{x}^T \ .$$

Example λ=(2,1), restrict to x_1, x_2, x_3.

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$$

Hence

$$\begin{aligned} s_{21}(x_1, x_2, x_3) &= x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 \\ &\quad +2x_1 x_2 x_3 \\ &= m_{21}(x_1, x_2, x_3) + 2m_{111}(x_1, x_2, x_3). \end{aligned}$$

It follows that $s_{21} = m_{21} + 2m_{111}.$

Note

$$s_\lambda(\mathbf{x}) = \sum_{T \in \mathrm{SSYT}(\lambda)} \mathbf{x}^T = \sum_{\alpha:(\alpha_1, \alpha_2, \ldots)} K_{\lambda\alpha} x_1^{\alpha_1} x_2^{\alpha_2} \cdots.$$

$K_{\lambda\alpha} = \#\{$SSYT of shape λ and content $\alpha\}$ = "Kostka number".

THEOREM *The Schur function s_λ is a symmetric function.*

To show: $K_{\lambda\alpha} = K_{\lambda\tilde{\alpha}}$

Sufficient: $\tilde{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{i-1}, \alpha_{i+1}, \alpha_i, \alpha_{i+2}, \ldots)$

Bijection: SSYT shape λ, content $\alpha \leftrightarrow$ SSYT shape λ, content $\tilde{\alpha}$

$$\begin{array}{ccccc}
 & & & & i \\
i & i & \underbrace{i \quad\quad i}_{r=2} & \underbrace{i+1 \quad i+1 \quad i+1 \quad i+1}_{s=4} & i+1 \\
i+1 & i+1 & & &
\end{array}$$

In each such row, convert the r i's and s $i+1$'s to s i's and r $i+1$'s:

$$\begin{array}{ccccc}
 & & & & i \\
i & i & \underbrace{i \quad i \quad i \quad i}_{s=4} & \underbrace{i+1 \quad i+1}_{r=2} & i+1 \\
i+1 & i+1 & & &
\end{array}$$

$\square$

Multiplying Schur functions:

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda$$

$c^\lambda_{\mu\nu}$: Littlewood-Richardson coefficient

Representation Theory, Algebraic Geometry, Linear Algebra

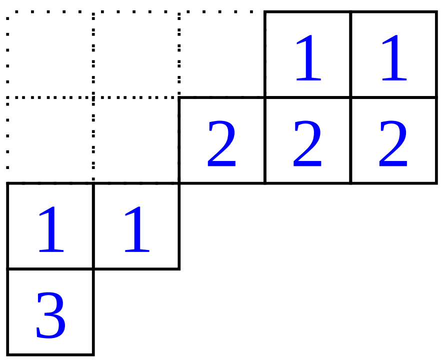
Examples

- $s_{21}s_{21} = s_{33} + s_{42} + s_{2211} + s_{222} + 2s_{321} + s_{411} + s_{3111}$
- $\lambda = (12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1)$
  $\mu = (8, 7, 6, 5, 4, 3, 2, 1)$
  $\nu = (8, 7, 6, 6, 5, 4, 3, 2, 1)$
  $c^\lambda_{\mu\nu} = 7869992$ (Anders Buch, John Stembridge)

Theorem *(Littlewood-Richardson, Schützenberger-Thomas)*
$c^\lambda_{\mu\nu} \in \mathbb{Z}$. *In fact,* $c^\lambda_{\mu\nu} \geq 0$.

Littlewood-Richardson Rule: $c^{\lambda}_{\mu\nu}$ = the number of SSYT of skew shape λ/μ and content ν whose reverse reading word is a ballot sequence.

EXAMPLE $\lambda = (5,5,2,1), \mu = (3,2), \nu = (4,3,1)$



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NOTE Since $s_\mu s_\nu = s_\nu s_\mu$, $c^{\lambda}_{\mu\nu} = c^{\lambda}_{\nu\mu}$. This is not at all obvious from the rule above.

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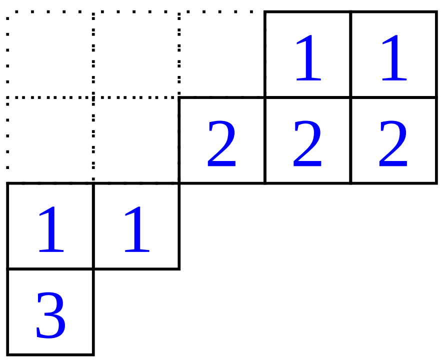

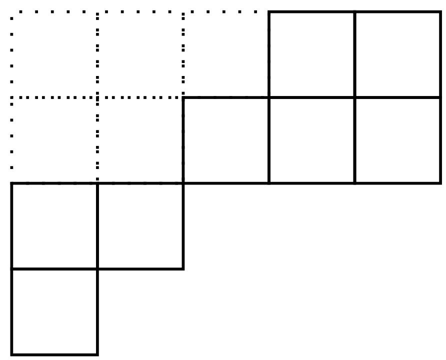

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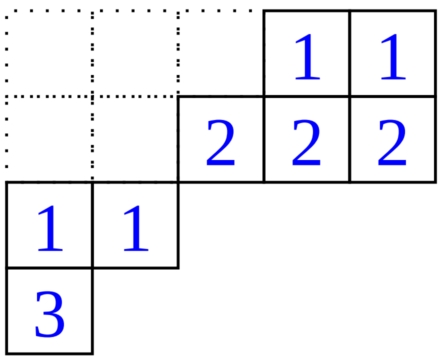

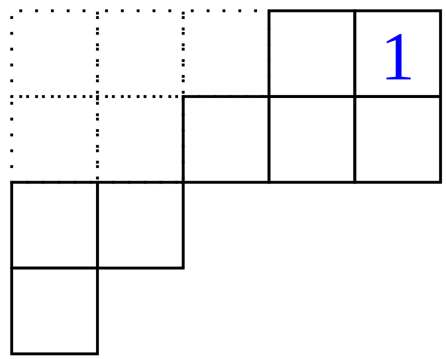

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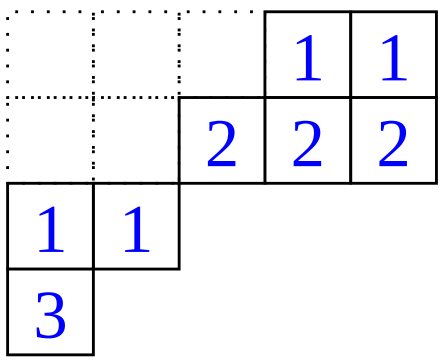

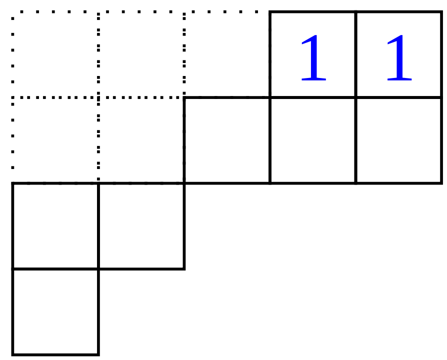

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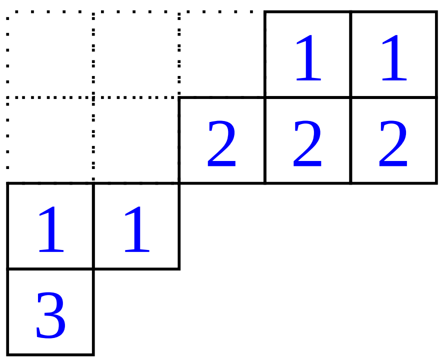

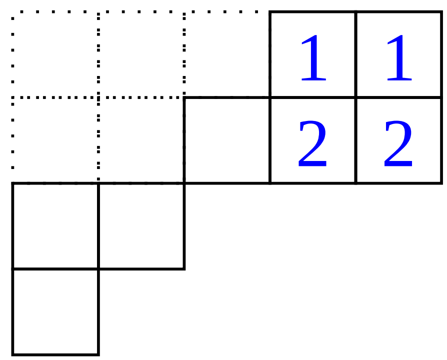

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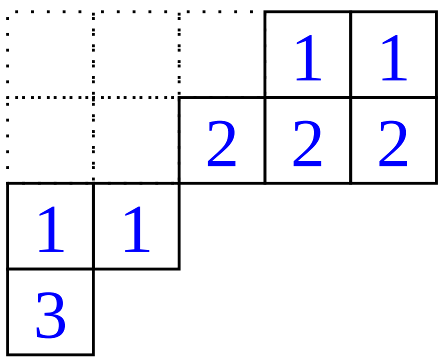
Example $\lambda = (5,5,2,1), \mu = (3,2), \nu = (4,3,1)$



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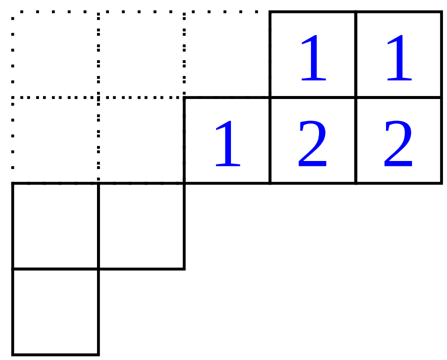

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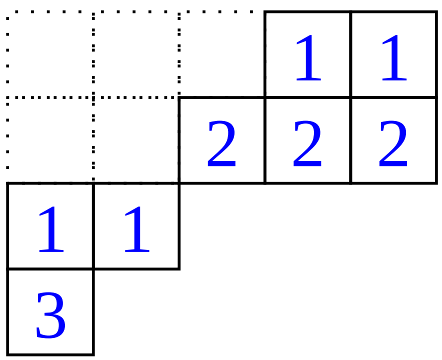

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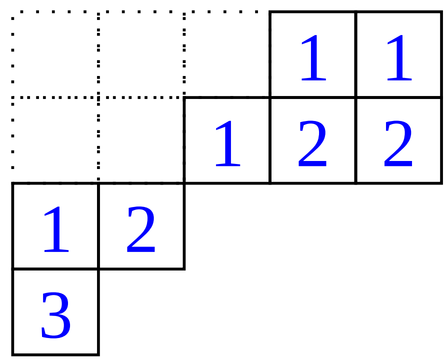

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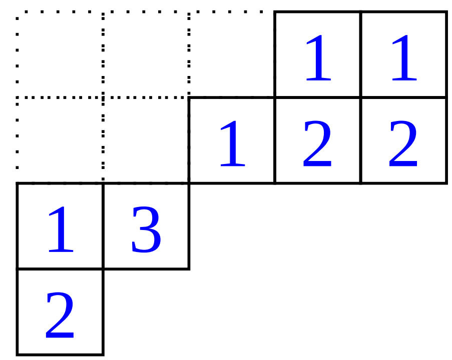

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NOTE Since $s_{\mu}s_{\nu} = s_{\nu}s_{\mu}$, $c^{\lambda}_{\mu\nu} = c^{\lambda}_{\nu\mu}$. This is not at all obvious from the rule above.

Since $c^{\lambda}_{\mu\nu} \geq 0$, we say that $s_\mu s_\nu$ is a Schur-positive or s-positive function.

QUESTION *When is*

$$s_\theta s_\phi - s_\mu s_\nu$$

s-positive?

CONJECTURE *(Fomin-Fulton-Li-Poon) For a pair (μ, ν) of partitions, let $\gamma : \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_{2p}$ be the decreasing rearrangement of the μ_i and ν_j's. Define two partitions*

$$\tilde{\mu} = (\gamma_1, \gamma_3, \ldots, \gamma_{2p-1}), \quad \tilde{\nu} = (\gamma_2, \gamma_4, \ldots, \gamma_{2p}).$$

Then $s_{\tilde{\mu}} s_{\tilde{\nu}} - s_\mu s_\nu$ is s-positive.

EXAMPLE $\mu = (5,1), \nu = (4,3,1,0)$

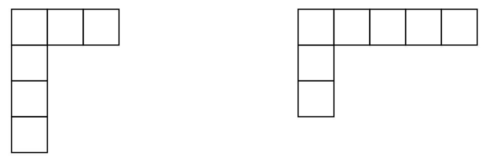
$$\begin{aligned} s_\mu s_\nu \quad = \quad & 2s_{743} + s_{752} + s_{7322} + 2s_{7331} + 3s_{7421} + s_{7511} + s_{54311} \\ & + s_{64211} + s_{74111} + s_{5432} + s_{5531} + s_{6332} + s_{6422} + 3s_{6431} \\ & + s_{6521} + s_{63311} + s_{73211} + s_{653} + s_{833} + 2s_{842} + s_{851} + s_{932} \\ & + s_{941} + 2s_{8321} + 2s_{8411} + s_{9311} + s_{5441} + s_{83111} + s_{644}. \end{aligned}$$

$\gamma = (5,4,3,1,1,0)$ so $\tilde{\mu} = (5,3,1), \tilde{\nu} = (4,1,0)$

$$\begin{aligned} s_{\tilde{\mu}} s_{\tilde{\nu}} - s_\mu s_\nu \quad = \quad & s_{752} + s_{7511} + s_{55211} + s_{65111} + s_{5522} + s_{5531} \\ & + 2s_{6521} + s_{653} + s_{761} + s_{6611} + s_{554} + s_{662}. \end{aligned}$$

Some special cases:

- True when $|\mu| + |\nu| \leq 35$, i.e. when $s_\mu s_\nu$ has degree ≤ 35.
- True when μ and ν are both hooks or both have just two rows.



- Fomin-Fulton-Li-Poon: If $c^\lambda_{\mu\nu} > 0$, then $c^\lambda_{\tilde{\mu}\tilde{\nu}} > 0$. In other words, the support of $s_\mu s_\nu$ is contained in the support of $s_{\tilde{\mu}} s_{\tilde{\nu}}$.

  QUESTION *(Fomin) Could it be true that if the support of $s_\mu s_\nu$ is contained in the support of $s_\theta s_\phi$, then $s_\theta s_\phi - s_\mu s_\nu$ is s-positive?*

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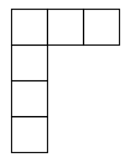 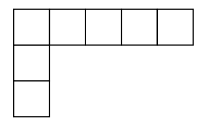

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  No. Buch, 4 hours later. Consider the coefficient of s_{5432} in $s_{3311} s_{321} - s_{4321} s_{211}$.

Stembridge: a classification of all pairs (μ, ν) such that $s_\mu s_\nu$ has all coefficients 0 or 1.

Fat hooks, near-rectangles, two-line rectangles, ...

Aside for linear algebra connection:

THEOREM*(Heckman 1982; Klyachko, 1998) Let α, β and γ be partitions with at most n rows. If the Saturation Conjecture is true, then TFAE:*

- $c^{\gamma}_{\alpha,\beta} > 0$
- *There exist $n \times n$ Hermitian matrices $A + B = C$ with eigenvalues α, β and γ.*

THEOREM*(Knutson and Tao, 1999) The Saturation Conjecture is true, i.e.*

$$c^{N\gamma}_{N\alpha,N\beta} > 0 \;\Rightarrow\; c^{\gamma}_{\alpha,\beta} > 0.$$

One last special case of Conjecture:

PROPOSITION *For a pair* (μ, ν) *of partitions, let* $\gamma : \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_{2p}$ *be the decreasing rearrangement of the* μ_i *and* ν_j*'s. As before, define*

$$\tilde{\mu} = (\gamma_1, \gamma_3, \ldots, \gamma_{2p-1}), \quad \tilde{\nu} = (\gamma_2, \gamma_4, \ldots, \gamma_{2p}).$$

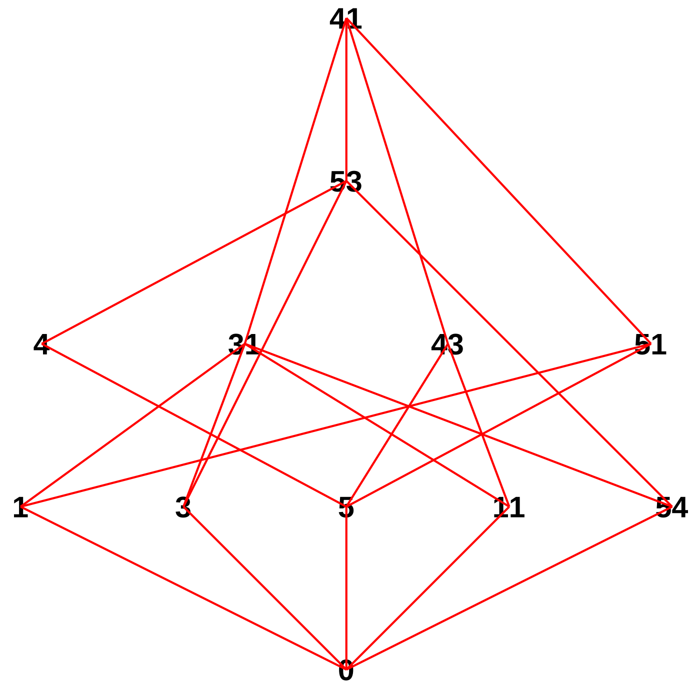
If $\{\mu_i, \nu_i\} = \{\gamma_{2i-1}, \gamma_{2i}\}$ *for all* i*, then* $s_{\tilde{\mu}} s_{\tilde{\nu}} - s_\mu s_\nu$ *is* s*-positive.*

Proof uses Jacobi-Trudi Identity, Plücker relations.

New perspective:

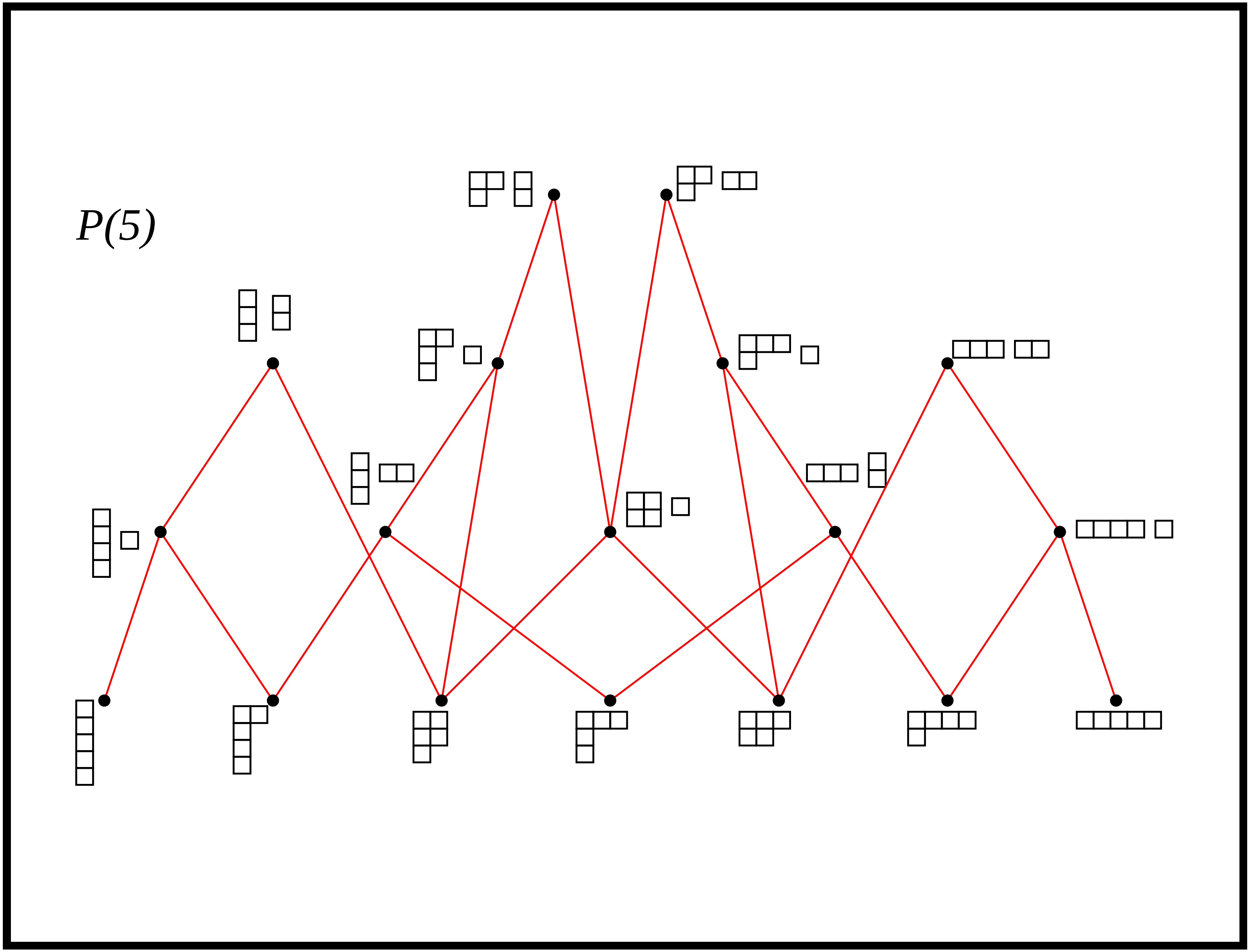
- Let $PD(\gamma)$ be the set of all possible "dealings" (μ, ν) of γ.
- Ordering of $PD(\gamma)$: $(\mu, \nu) \leq (\theta, \phi)$ if $s_\theta s_\phi - s_\mu s_\nu$ is s-positive.
- $PD(\gamma)$ is then a partially ordered set (poset).

EXAMPLE $PD(5,4,3,1,1)$:



CONJECTURE *(FFLP reformulated) $PD(\gamma)$ has a top element, namely $(\tilde{\mu}, \tilde{\nu})$.*

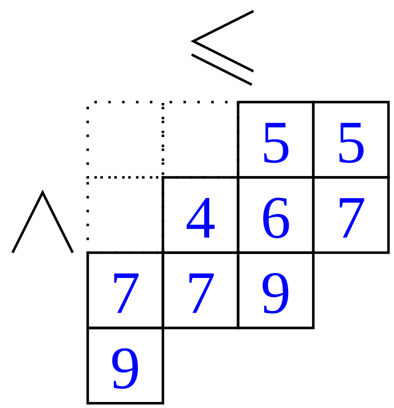
BIG QUESTION *Fix n. Consider all pairs of partitions (μ, ν) satisfying $|\mu| + |\nu| = n$. Order them in the same way to form a poset $P(n)$. What can we say about $P(n)$?*


P(5)

Finish with two more conjectures...

Recall: SSYT of skew shape μ/α, SSYT(μ/α):

T:



We define skew Schur functions in the analogous way:

$$s_{\mu/\alpha}(\mathbf{x}) = \sum_{T \in \mathrm{SSYT}(\mu/\alpha)} \mathbf{x}^T \, .$$

- Symmetric: exactly the same argument as before
- Lots of these: 254,777 skew shapes with 12 boxes
- Example: $s_{4431/21} = s_{3321} + s_{4221} + s_{4311} + s_{333} + 2s_{432} + s_{441}$

- Suppose $\alpha \subseteq \mu$, $\beta \subseteq \nu$.
- Construct $(\tilde{\mu}, \tilde{\nu})$ from (μ, ν).
- Construct $(\tilde{\alpha}, \tilde{\beta})$ from (α, β).
- Easy to show: $\tilde{\alpha} \subseteq \tilde{\mu}$, $\tilde{\beta} \subseteq \tilde{\nu}$.

We get the following skew-shape analogue of FFLP's Conjecture:

Conjecture $s_{\tilde{\mu}/\tilde{\alpha}} s_{\tilde{\nu}/\tilde{\beta}} - s_{\mu/\alpha} s_{\nu/\beta}$ *is s-positive.*

Checked for $|\mu/\alpha| + |\nu/\beta| \leq 12$. (966,137 pairs of skew shapes).

Recall our original question: starting with any pair (μ, ν), how can we define (θ, ϕ) to get

$$s_\theta s_\phi - s_\mu s_\nu$$

s-positive?

For geometrical reasons, Fomin, Fulton, Li & Poon suggest the following:

CONJECTURE *Given an ordered pair (μ, ν) of partitions with the same number of parts, define a new ordered pair (μ^*, ν^*) by the following recipe:*

$$\begin{aligned} \mu_k^* &= \mu_k - k + \#\{l \mid \nu_l - l \geq \mu_k - k\}; \\ \nu_l^* &= \nu_l - l + 1 + \#\{k \mid \mu_k - k > \nu_l - l\}. \end{aligned}$$

Then $s_{\mu^*} s_{\nu^*} - s_\mu s_\nu$ is s-positive.

See `www.arxiv.org/math.AG/0301307` for more details.