The Schur-Positivity Poset

Peter McNamara

Bucknell University

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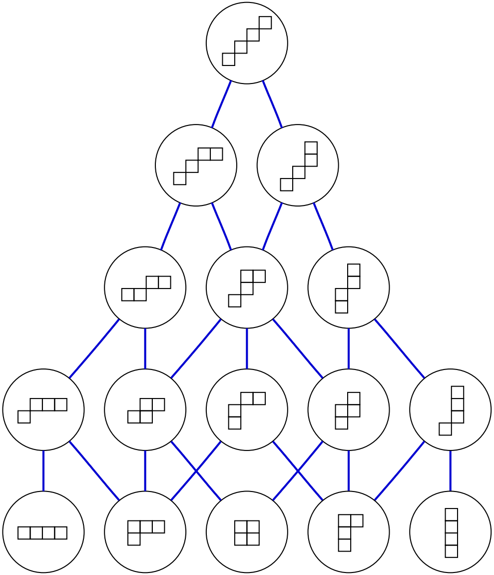
Slides and papers available from
www.facstaff.bucknell.edu/pm040/

Outline

- Introduction to the Schur-positivity poset
- Some known properties
- Some unknown properties
- Focus on necessary conditions for $A \leq_s B$.

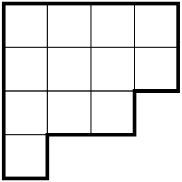
Preview

$n = 4$



Schur functions

- Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$
- Young diagram.
  Example:
  $\lambda \quad = (4, 4, 3, 1)$



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$\leqslant$

$\wedge$

1	3	3	4
4	4	4	9
5	6	6	
7			

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The Schur function s_λ in the variables $x = (x_1, x_2, \ldots)$ is then defined by

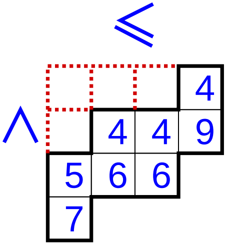
$$s_\lambda \quad = \sum_{\text{SSYT } T} x_1^{\#\text{1's in } T} x_2^{\#\text{2's in } T} \cdots .$$

Example.

$s_{4431} \quad = x_1 x_3^2 x_4^4 x_5 x_6^2 x_7 x_9 + \cdots .$

Skew Schur functions

- Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$
- μ fits inside λ.
- Young diagram.
  Example:
  $\lambda/\mu = (4,4,3,1)/(3,1)$
- Semistandard Young tableau (SSYT)



The skew Schur function $s_{\lambda/\mu}$ in the variables $x = (x_1, x_2, \ldots)$ is then defined by

$$s_{\lambda/\mu} = \sum_{\text{SSYT } T} x_1^{\#1\text{'s in } T} x_2^{\#2\text{'s in } T} \cdots .$$

Example.

$$s_{4431/31} = \qquad x_4^3 x_5 x_6^2 x_7 x_9 + \cdots .$$

Some reasons why we care

- Skew Schur functions are symmetric in the variables $x = (x_1, x_2, \ldots)$.
- The Schur functions form a basis for the algebra of symmetric functions (over $\mathbb{Q}$, say).
- Connections to algebraic geometry (Schubert calculus), representation theory (S_n, $GL(n, \mathbb{C})$).

Littlewood-Richardson Rule

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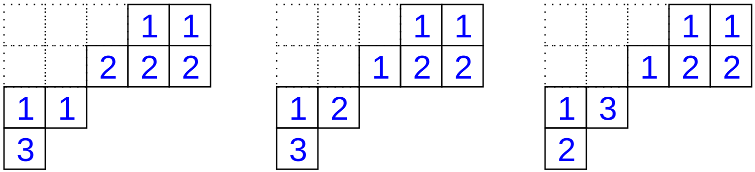
Example.

When $\lambda = (5, 5, 2, 1), \mu = (3, 2), \nu = (4, 3, 1)$, we get $c^{\lambda}_{\mu\nu} = 2$.



11222113 No   11221213 Yes   11221312 Yes

Main definitions

Key point: $c^{\lambda}_{\mu\nu} \geq 0$.

$s_{\lambda/\mu}$ is Schur-positive, i.e. coefficients in Schur expansion are all non-negative.

Natural connections between Schur-positivity and representation theory.

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When is $s_{\lambda/\mu} - s_{\sigma/\tau}$ Schur-positive?

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Definition.
Let A, B be skew shapes. We say that

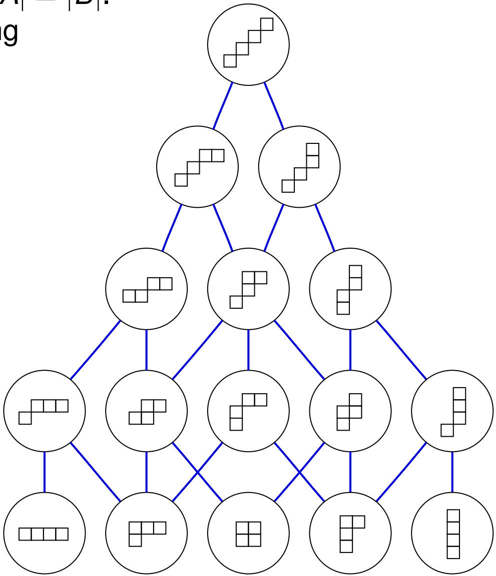
$$A \leq_s B \quad \text{if} \quad s_B - s_A$$

is Schur-positive.

Goal: Characterize the Schur-positivity order $\leq_s$ in terms of skew shapes.

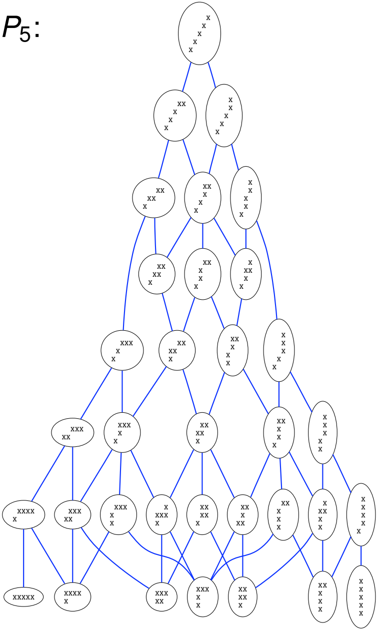
Example of a Schur-positivity poset

If $A \leq_s B$ then $|A| = |B|$.
Call the resulting ordered set P_n.
Then P_4:

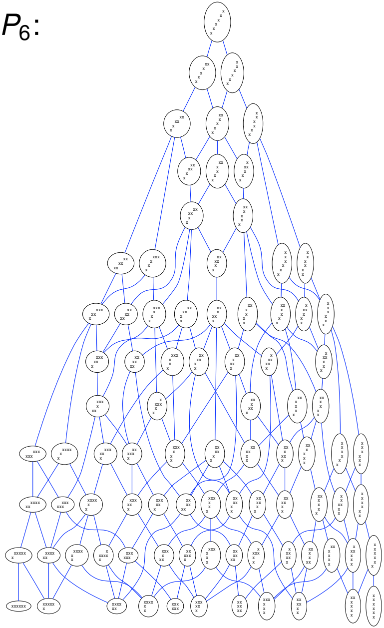


More examples

P_5:

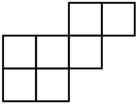
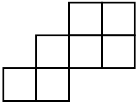


P_6:

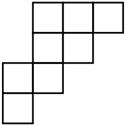
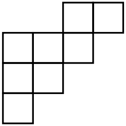


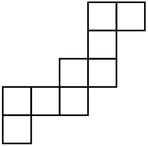
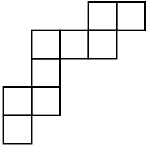
Known properties: first things first

$\leq_s$ is not yet anti-symmetric. So identify skew shapes such as

1.  $=$ 

[EC2, Exercise 7.56(a), 2-]

2.  $=$ 

3.  $=$ 

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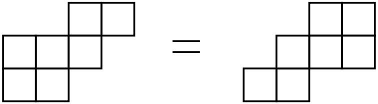

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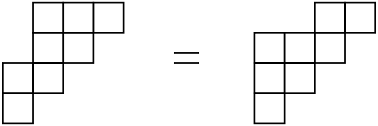

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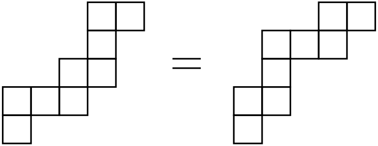

Definition.
A ribbon is a connected skew shape containing no 2×2 rectangle.

Known properties: skew Schur equality

Question: When is $s_A = s_B$?

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  Complete classification of equality of ribbon Schur functions
- Vic Reiner, Kristin Shaw, Stephanie van Willigenburg (2006)
- McN., Stephanie van Willigenburg (2006)

Enough for our purposes: we can consider P_n to be a poset.

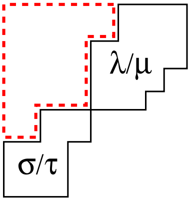
Open Problem: Find necessary and sufficient conditions on A and B for $s_A = s_B$.

Known properties: Sufficient conditions

Sufficient conditions for $A \leq_s B$:

- Alain Lascoux, Bernard Leclerc, Jean-Yves Thibon (1997)
- Andrei Okounkov (1997)
- Sergey Fomin, William Fulton, Chi-Kwong Li, Yiu-Tung Poon (2003)
- Anatol N. Kirillov (2004)
- Thomas Lam, Alex Postnikov, Pavlo Pylyavskyy (2005)
- François Bergeron, Riccardo Biagioli, Mercedes Rosas (2006)
- ...

Note: $s_{\lambda/\mu}s_{\sigma/\tau}$ is a special case of s_A.

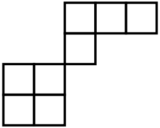


Lam, Postnikov and Pylyavskyy's result

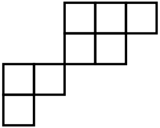
Theorem [LPP]. For skew shapes λ/μ and σ/τ,

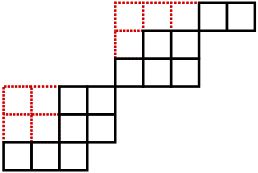
$$s_{\lambda/\mu} s_{\sigma/\tau} \leq_s s_{\lambda\cup\sigma/\mu\cup\tau} s_{\lambda\cap\sigma/\mu\cap\tau}$$

Examples.



$\leq_s$





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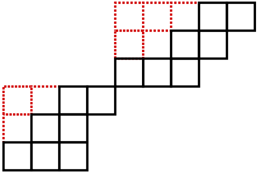


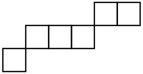
Known properties: special classes of skew shapes

Notation. Write $\lambda \preccurlyeq \mu$ if λ is less than or equal to μ in dominance order, i.e.

$$\lambda_1 + \cdots \lambda_i \leq \mu_1 + \cdots \mu_i \text{ for all } i.$$

- Macdonald's "Symmetric functions and Hall polynomials": For horizontal strips, $A \leq_s B$ if and only if

$$\text{row lengths of } A \succcurlyeq \text{row lengths of } B$$



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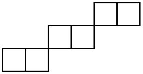

P_n restricted to horizontal strips: (dual of the) dominance lattice.

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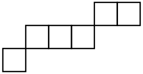
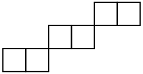
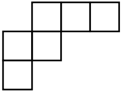
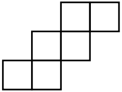
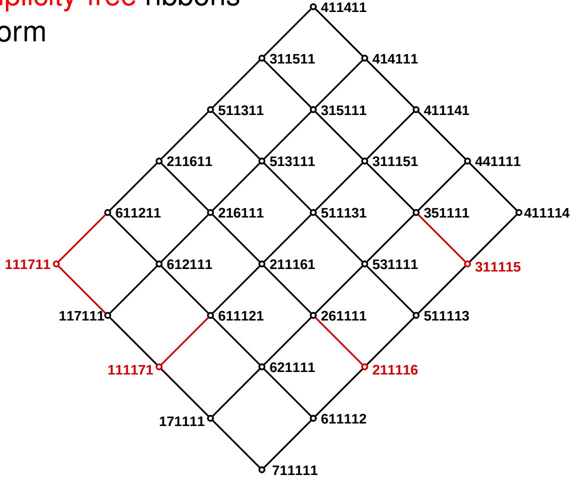
   $\leq_s$ 

  P_n restricted to horizontal strips: (dual of the) dominance lattice.

- Ron King, Trevor Welsh, Stephanie van Willigenburg (2007): For ribbons with decreasing row lengths and equal numbers of rows, same is true.

  321 =  $\leq_s$  = 222

- McN., Stephanie van Willigenburg (2007): For a given number of boxes and rows,
  the poset of multiplicity-free ribbons
  is always of the form

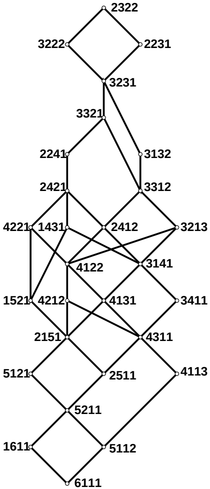


Unknown properties: general ribbons

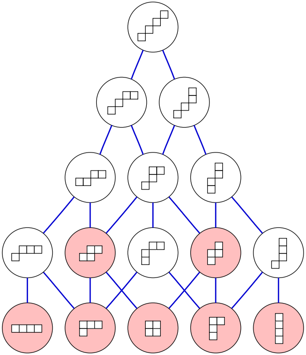
Open Problem: Explain the Schur-positivity order for general ribbons.

Suffices to fix #boxes and #rows.

Ribbons with 9 boxes and 4 rows:



Unknown properties: maximal connected skew shapes



Question: What are the maximal elements of P_n among the connected skew shapes?

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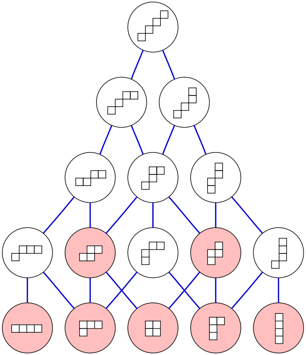

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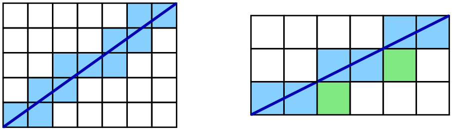
Conjecture [McN., Pylyavskyy]. For each $r = 1, \ldots, n$, there is a unique maximal connected element with r rows, namely the ribbon marked out by the diagonal of an r-by-$(n-r+1)$ box.

Examples.



Necessary conditions

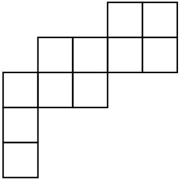
Question: Suppose $A \leq_s B$ (i.e. $s_B - s_A$ is Schur-positive).
Then what can we say about the shapes A and B?

Such necessary conditions for $A \leq_s B$ give us a way to show that $C \not\leq_s D$.

Example. If $A \leq_s B$, then $|A| = |B|$.

Notation. For a skew shape A, let $\text{rows}(A)$ denote the partition of row lengths of A. Define $\text{cols}(A)$ similarly.

Example. $\text{rows}(A) = 43211$, $\text{cols}(A) = 32222$.



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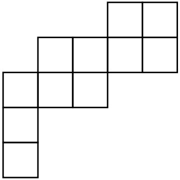
$$\begin{aligned} s_A &= s_{551} + s_{542} + 2s_{5411} + s_{533} + 2s_{5321} + s_{53111} \\ &\quad + s_{52211} + s_{4421} + s_{44111} + s_{4331} + s_{43211}. \\ \mathrm{support}(A) &= \{551, 542, 5411, 533, 5321, 53111, \\ &\quad 52211, 4421, 44111, 4331, 43211\}. \end{aligned}$$



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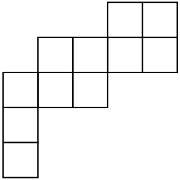
Proposition. In the Schur expansion of A:

- rows(A) is the **least** dominant partition in the support of A.
- $(\text{cols}(A))^t$ is the **most** dominant partition in the support of A.



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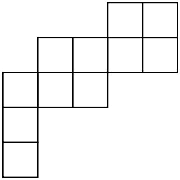

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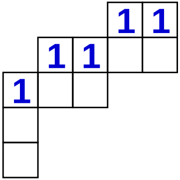

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			1	1
	1	1	**2**	**2**
1	**2**	**2**		
2				

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$$\begin{aligned} s_A &= s_{551} + s_{542} + 2s_{5411} + s_{533} + 2s_{5321} + s_{53111} \\ &\quad + s_{52211} + s_{4421} + s_{44111} + s_{4331} + s_{43211}. \\ \text{support}(A) &= \{551, 542, 5411, 533, 5321, 53111, \\ &\quad 52211, 4421, 44111, 4331, 43211\}. \end{aligned}$$

Proposition. In the Schur expansion of A:

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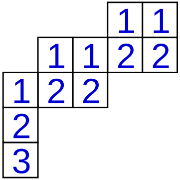

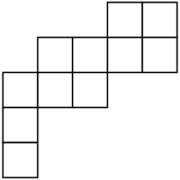

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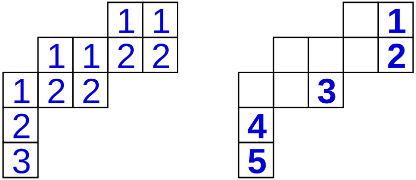

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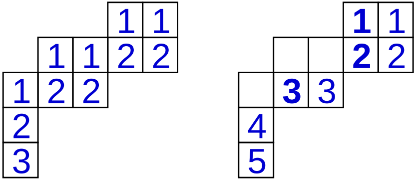

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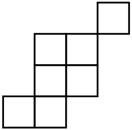
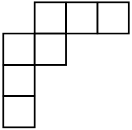
$C =$  $D =$ 

$\text{rows}(C) = 2221 \prec 3211 = \text{rows}(D)$.

Thus $C \not\leq_s D$.

Key definitions: generalize rows(A) and cols(A)

Definitions [Reiner, Shaw, van Willigenburg]. For a skew shape A, let $\mathrm{overlap}_k(i)$ be the number of columns occupied in common by rows $i, i+1, \ldots, i+k-1$.
Then $\mathrm{rows}_k(A)$ is the weakly decreasing rearrangment of $(\mathrm{overlap}_k(1), \mathrm{overlap}_k(2), \ldots)$.
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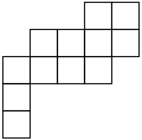
$A =$ 

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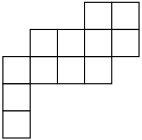
$A =$ 

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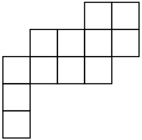
$A =$ 

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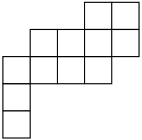
$A =$ 

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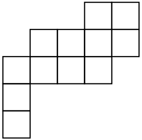
$A =$ 

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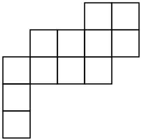
$A =$ 

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- $\mathrm{cols}_1(A) = \mathrm{cols}(A) = 33222$, $\mathrm{cols}_2(A) = 2221$, $\mathrm{cols}_3(A) = 211$, $\mathrm{cols}_4(A) = 11$, $\mathrm{cols}_k(A) = \emptyset$ for $k > 4$.

Necessary conditions

Theorem [RSvW]. Let A and B be skew shapes. If $s_A = s_B$, then

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Corollary. Let A and B be skew shapes. If $\mathrm{support}(A) = \mathrm{support}(B)$, then

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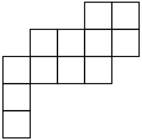
Relating $\mathrm{rows}_k(A)$ and $\mathrm{cols}_k(A)$

Let $\mathrm{rects}_{k,\ell}(A)$ denote the number of $k \times \ell$ rectangular subdiagrams contained inside A.

$A =$  $\mathrm{rects}_{3,1}(A) = 2$, $\mathrm{rects}_{2,2}(A) = 3$, etc.

Theorem [RSvW]. Let A and B be skew shapes. TFAE:

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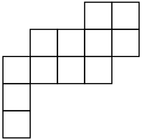
$A =$  $\mathrm{rects}_{3,1}(A) = 2$, $\mathrm{rects}_{2,2}(A) = 3$, etc.

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Summary result

Theorem [McN]. Let A and B be skew shapes. If $A \leq_s B$, i.e. $s_B - s_A$ is Schur-positive, or if A and B satisfy the weaker condition that $\mathrm{support}(A) \subseteq \mathrm{support}(B)$, then the following three equivalent sets of conditions are true:

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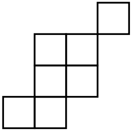
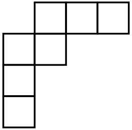
$C =$  $D =$ 

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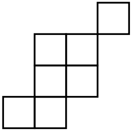
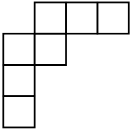
$C =$  $D =$ 

$\text{rows}(C) = 2221 \prec 3211 = \text{rows}(D)$. Thus $C \not\leq_s D$.

$\text{rows}_2(C) = 21 \succ 111 = \text{rows}_2(D)$. Thus $D \not\leq_s C$.

Outlook

- Instead of looking at the Schur-positivity poset, could look at the support containment poset; it seems to have more structure.
- Almost nothing is known about the covering relations in P_n.
- Why restrict to skew Schur functions? Could try:
  - Stanley symmetric functions
  - Hall-Littlewood polynomials
  - LLT-polynomials
  - Cylindric Schur functions
  - Skew Grothendieck polynomials
  - Poset quasisymmetric functions
  - Wave Schur functions
  - ...