Quasisymmetric functions distinguishing trees

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Joint work with:
Jean-Christophe Aval
LaBRI, CNRS, Université de Bordeaux

Karimatou Djenabou
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Enumerative and Algebraic Combinatorics
SaganFest
25 February 2024

Slides and paper available from

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Coauthor ratio: 0.939

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Outline

- Chromatic (quasi)symmetric functions and the motivating conjectures
- Converting to a poset question; more conjectures
- Some old and new results; one last conjecture

The chromatic polynomial

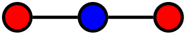
George Birkhoff, 1912

Graph $G = (V, E)$

Coloring: a map $\kappa : V \to \{1, 2, 3, \dots\}$

Proper coloring: adjacent vertices get different colors.

Proper



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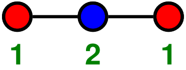

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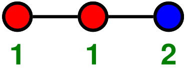

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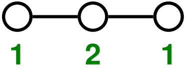

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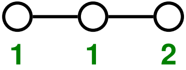

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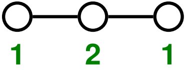

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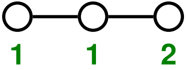


Chromatic polynomial: $\chi_G(k)$ is the number of proper colorings of G when k colors are available.

Example. If T is any tree with n vertices, $\chi_T(k) = k(k-1)^{n-1}$.

The chromatic symmetric function

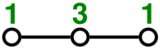
Richard Stanley, 1995

Graph $G = (V, E)$

$V = \{v_1, v_2, \ldots, v_n\}$

To a proper coloring κ, we associate the monomial in commuting variables $x_1, x_2, \ldots$

$$x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_n)}.$$



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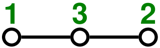

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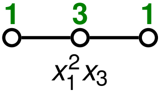

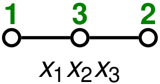

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- $X_G(\mathbf{x})$ is a symmetric function
- Setting $x_i = 1$ for $1 \le i \le k$ and $x_i = 0$ otherwise yields $\chi_G(k)$.

Can $X_G(\mathbf{x})$ distinguish graphs?

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[Aliniaeifard, Aliste-Prieto, Crew, Dahhberg, de Mier, Fougere, Heil, Ji, Loebl, Loehr, Martin, Morin, Orellana, Scott, Smith, Sereni, Spirkl, Tian, Wagner, Wang, Warrington, van Willigenburg, Zamora, ...]

The Loehr–Warrington Conjecture

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(Surprising) Conjecture 2 (Nick Loehr & Greg Warrington, 2022). $X_G(1, q, q^2, \ldots, q^{n-1})$ distinguishes trees with n vertices, i.e. if T and U are non-isomorphic trees with n vertices, then

$$X_T(1, q, q^2, \ldots, q^{n-1}) \neq X_U(1, q, q^2, \ldots, q^{n-1}).$$

Why surprising?

- $X_T(1, q, q^2, \ldots, q^{n-1})$ is a polynomial in one variable!
- Compare to $X_G(\mathbf{x})$ and $\chi_G(k)$.

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- $X_T(1, q, q^2, \ldots, q^{n-1})$ is a polynomial in one variable!
- Compare to $X_G(\mathbf{x})$ and $\chi_G(k)$.
- The data suggests that fewer than n nonzero variables suffice.

The chromatic quasisymmetric function

John Shareshian & Michelle Wachs, 2014; Brittney Ellzey, 2017.

Directed graph $\overrightarrow{G} = (V, E)$.

Ascent of proper coloring κ: directed edge $u \to v$ with $\kappa(u) < \kappa(v)$

asc(κ): the number of ascents of κ.

Example. Colors $a < b < c$

$\kappa(v_1)$	$\kappa(v_2)$	$\kappa(v_3)$	asc(κ)
a	b	c	1
a	c	b	2
b	a	c	0
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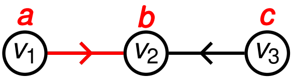

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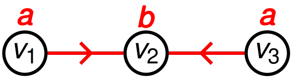

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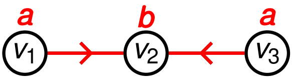

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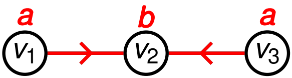

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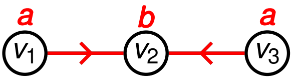

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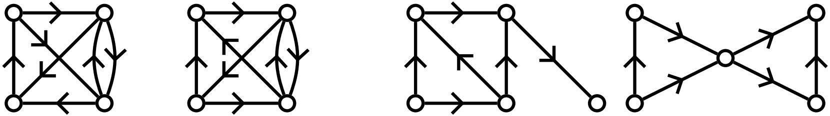

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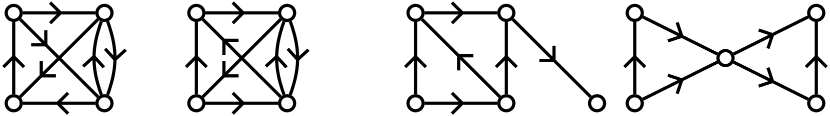

Conjecture 3 (ADM; stated as a question by Per Alexandersson and Robin Sulzgruber, 2021).

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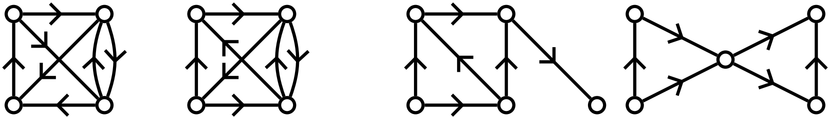

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This conjecture was our original goal. **Strategy: translate to posets.**

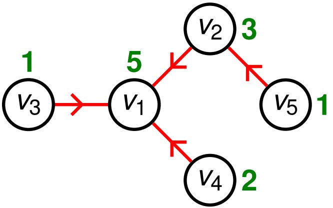
Translating to posets

$$X_{\vec{G}}(\mathbf{x}, t) = \sum_{\text{proper } \kappa} t^{\text{asc}(\kappa)} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_n)}.$$

Want to show: $X_{\vec{T}}(\mathbf{x}, t) \neq X_{\vec{U}}(\mathbf{x}, t)$.

Key insight:

- Look at the coefficient of the highest power of t.
- It's enough to show these coefficients are different for $\vec{T}$ and $\vec{U}$.
- So just look at colorings where all edges are ascents.



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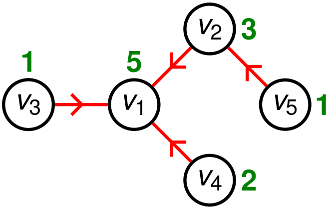

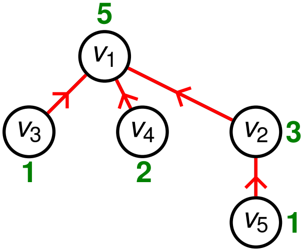

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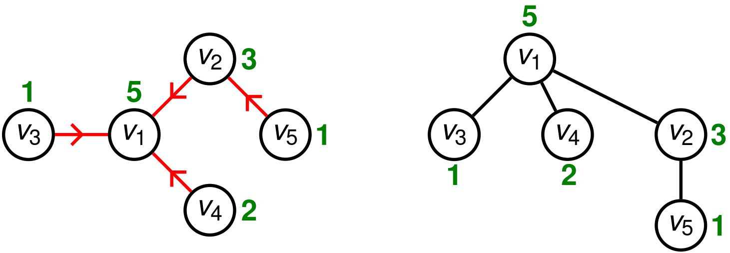

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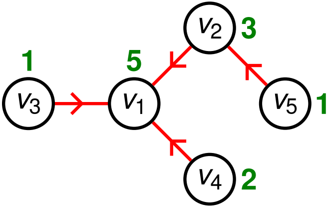
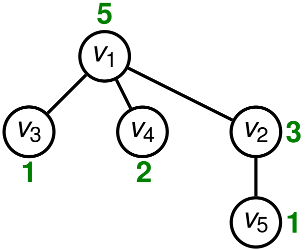
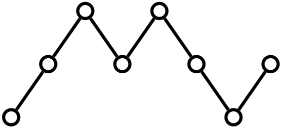
- The corresponding coloring is a strict P-partition (strictly order-presevering map)





Two nice examples

Example. If $\overrightarrow{G}$ is a directed path, we get a fence poset.
[Sagan, Elizalde, Kantarci Oğuz, McConville, Plante, Ravichandran, Roby, Smyth, ...]
Conjecture still open in this case in full generality (?)



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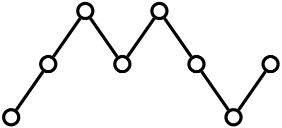

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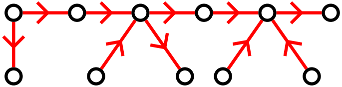

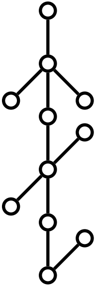

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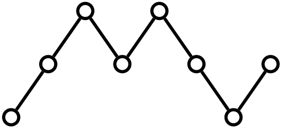

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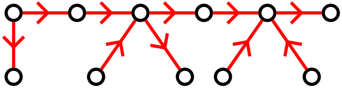
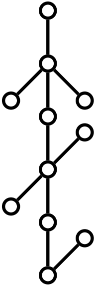




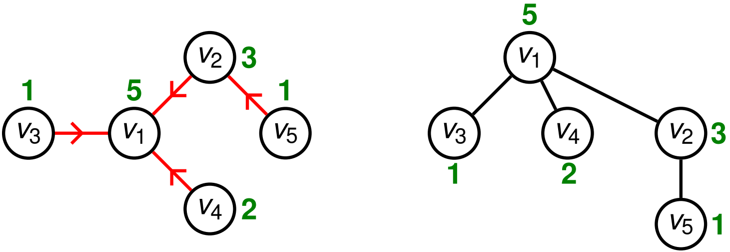
Propostion (Nate Lesnevich & M., 2022).
$X_{\overrightarrow{G}}(\mathbf{x}, t)$ distinguishes these caterpillar digraphs.

Translating to posets

$$X_{\vec{G}}(\mathbf{x}, t) = \sum_{\text{proper } \kappa} t^{\text{asc}(\kappa)} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_n)}.$$

The leading coefficient is the strict P-parition enumerator:

$$K_P^{<}(\mathbf{x}) = \sum_{\text{strict } P\text{-partition } f} x_{f(p_1)} x_{f(p_2)} \cdots x_{f(p_n)}.$$



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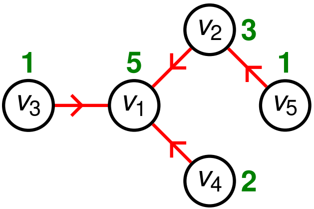

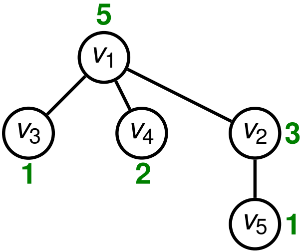

Project. Study equality among $K_P^{<}(\mathbf{x})$.
[Browning, Féray, Hasebe, Hopkins, Kelly, Lesnevich, Liu, M., Tsujie, Ward, Weselcouch, ...]

Can $K_{(P,\omega)}(\mathbf{x})$ distinguish posets?

Conjecture 4 (ADM; stated as a question by Takahiro Hasebe and Shuhei Tsujie, 2017).

$K_P^{<}(\mathbf{x})$ distinguishes posets that are trees.
i.e. if tree posets P and Q are not isomorphic, then $K_P^{<}(\mathbf{x}) \neq K_Q^{<}(\mathbf{x})$.

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Theorem (Hasebe & Tsujie, 2017).

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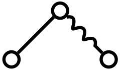
Stanley's (P, ω)-partitions: both strict and weak edges, i.e., labeled posets.

f only needs to weakly increase along weak (springy) edges.



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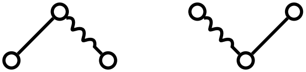

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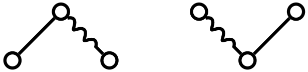

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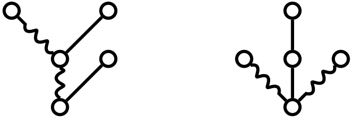

Definition. A labeled poset that is a rooted tree is said to be a fair tree if for each vertex, its outgoing edges up to its children are either all strict or all weak.

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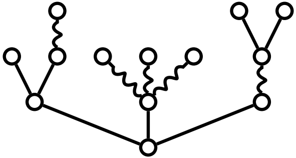

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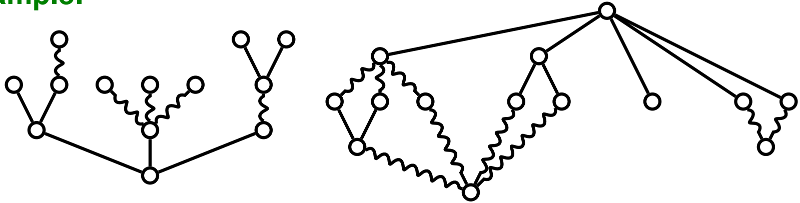

Definition. More generally, we define the set $\mathcal{C}$ of labeled posets recursively by:

1. the one-element labeled poset $[1]$ is in $\mathcal{C}$;
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3. $\mathcal{C}$ is closed under the ordinal sums $(P,\omega) \wr [1]$ and $(P,\omega) \uparrow [1]$;
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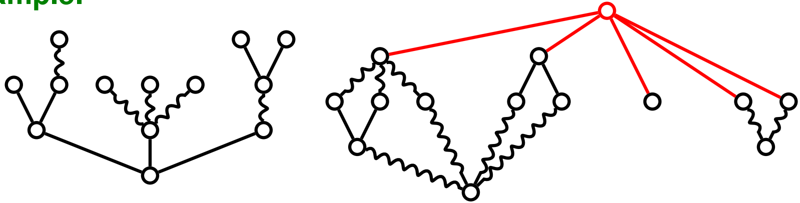

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Theorem [ADM, 2022].
$K_{(P,\omega)}(\mathbf{x})$ distinguishes elements of $\mathcal{C}$, so in particular fair trees.

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Irreducibility is also the crux for

- Hasebe & Tsujie;
- Ricki Ini Liu & Michael Weselcouch ($K_P^{<}(\mathbf{x})$ distinguishes series-parallel posets; includes irreducibility for general connected P with all strict edges, 2020).

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Remark. This specialization has a nice interpretation for $K_{(P,\omega)}$: if

$$K_{(P,\omega)}(1, q, q^2, \ldots, q^{k-1}) = \sum_{N \geq 0} a(N) q^N,$$

then we see that $a(N)$ counts the number of (P, ω)-partitions $f : P \to \{0, \ldots, k-1\}$ of N.

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Thanks for your attention!

Happy Birthday Bruce!