The shortest path poset of finite Coxeter Groups

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Coxeter Groups

Groups with presentation

$$\langle S \mid (ss')^{m(s,s')} = e, \text{ for all } s, s' \in S \rangle$$

where

- $m(s, s) = 1$
- $m(s, s') = m(s', s) \geq 2$ for $s \neq s'$
- $m(s, s') = \infty$ means that there is no relation between $s$ and $s'$. 
Examples.

- $\mathbb{Z}_2 = \langle s \mid s^2 = 1 \rangle$.

- The dihedral group of order $2m$.
  \[ l_2(m) = \langle s_1, s_2 \mid (s_1 s_2)^m = (s_2 s_1)^m = s_1^2 = s_2^2 = 1 \rangle. \]
  When $m \geq 3$, this is the group of symmetries of the $m$-gon.

- The symmetric group.
  \[ A_{n-1} = S_n = \langle s_1, s_2, \ldots, s_{n-1} \mid (s_i s_j)^{m(s_i, s_j)} \rangle, \]
  where $s_i = (i, i + 1)$, $m(s_i, s_{i+1}) = 3$ and otherwise $m(s_i, s_j) = 2$ for $i < j$. 
Basic Definitions

- Each \( w \in W \) can be expressed as \( w = s_1 s_2 \ldots s_n \) with \( s_i \in S \). If \( n \) is minimal, then \( s_1 s_2 \ldots s_n \) is a reduced expression for \( w \). In this case, we define the length function by \( \ell(w) = n \).

- \( T(W) = \{ wsw^{-1} \mid w \in W, s \in S \} \) is the set of reflections of \((W, S)\).

- Bruhat Order: Let \( v, w \in W \). We say that \( v \leq w \) if and only if there exist \( t_1, \ldots, t_k \in T \) so that \( vt_1 t_2 \cdots t_k = w \) with \( \ell(vt_1) > \ell(v) \) and \( \ell(vt_1 \cdots t_i) > \ell(vt_1 \cdots t_{i-1}) \) for \( i > 1 \).

- If \( W \) is finite, then there exists a maximal-length word \( w_0^W \); that is, \( \ell(w) \leq \ell(w_0^W) \) for all \( w \in W \).

- If \( |W| < \infty \), then \( \ell(w_0^W) = |T(W)|. \)
The directed graph \((V, E)\) consisting of \(V = W\) and \((u, v) \in E\) if \(\ell(u) < \ell(v)\) and there exists \(t \in T\) with \(ut = v\) is called the Bruhat graph.

For example, consider \(S_3\) with generators \(s_1 = (1, 2), s_2 = (2, 3)\), with labeling \(1 \rightarrow s_1, 2 \rightarrow s_1s_2s_1, 3 \rightarrow s_2\).
A reflection order is a total order $<_T$ on the reflections of $W$ so that for any dihedral reflection subgroup $W'$ (i.e., $W'$ has two generators, $x, y \in T$), then either

$$x <_T xyx <_T xyxyx <_T \ldots <_T yxyxy <_T yxy <_T y$$

or

$$y <_T yxy <_T yxyxy <_T \ldots <_T xyxyx <_T xyx <_T x$$

where $x$ and $y$ are the generators of $W'$. 
Complete \textbf{cd-index}

Fix a reflection ordering \( <_T \). Consider a chain (path) \( C \) in the Bruhat graph of \([u, v]\) labeled by reflections, say

\[
C = (t_1, t_2, \ldots, t_k)
\]

The descent set of \( C \) is

\[
D(C) = \{ i \in [k - 1] \mid t_{i+1} <_T t_i \}
\]

The \textbf{complete cd-index} encodes the descent sets of all the Bruhat paths.
Complete **cd**-index

The encoding is done as follows: Let $\Delta = (t_1, t_2, \ldots, t_k)$ be a path of length $k$ from $u$ to $v$. Then define $w(\Delta) = x_1x_2 \cdots x_{k-1}$ where

$$x_i = \begin{cases} a & \text{if } t_i <_T t_{i+1} \text{ (for ascent)} \\ b & \text{if } t_{i+1} <_T t_i \end{cases}$$

Now consider the polynomial $\sum_{\Delta} w(\Delta)$. Set

$$c = a + b$$
$$d = ab + ba$$

After the substitution, $\sum_{\Delta} w(\Delta)$ becomes a polynomial with variables $c$ and $d$. This is denoted by $\tilde{\psi}_{u,v}$, and it is called the complete **cd**-index of $[u, v]$. 
Example

Consider $S_3$ with generators $s_1 = (1, 2)$ and $s_2 = (2, 3)$, and reflection ordering $s_1 = (1, 2) < T s_1 s_2 s_1 = (1, 3) < T s_2 = (2, 3)$.

\[
\begin{array}{c}
S_1 < T S_1 S_2 S_1 < T S_2 \\
123 & a^2 \\
131 & ab \\
313 & ba \\
321 & b^2 \\
\hline
2 & 1 \\
\end{array}
\]

\[
\tilde{\psi}_{e, s_1 s_2 s_1} = c^2 + 1
\]
A bigger example

\[ \tilde{\psi}_{12435,53142} = c^5 + 6cdc^2 + 6c^2dc + 3dc^3 + 3c^3d + 7cd^2 + 7d^2c + 6dcd + c^3 + 2dc + 2cd \]
Shortest Path Poset of $W$

If $W$ is a finite Coxeter group, we can form a poset $SP(W)$ with the shortest paths of $W$. For example, consider the Bruhat graph of $B_2$ (signed permutations of two elements)
$SP(W)$ is a gra

The absolute length of $w \in W$ is the minimal number of reflections $t_1, \ldots, t_k$ so that $t_1 t_2 \cdots t_k = w$. We write $\ell_T(w) = k$. 

Bruhat Order for $A_2$  

Absolute Order for $A_2$
$SP(A_{n-1})$

How to describe the shortest paths from $e$ to $w_{0}^{A_{n-1}} = n\ n\ n - 1 \ldots 2\ 1$?

Let $r_{i} = (i\ n + 1 - i)$ and $k = \left\lfloor \frac{n}{2} \right\rfloor$. Then

**Theorem**

If $t_{1} t_{2} \cdots t_{k} = w_{0}^{A_{n-1}}$ then

- $\{t_{1}, t_{2}, \ldots, t_{k}\} = \{r_{1}, r_{2}, \ldots, r_{k}\}$
- $t_{i} t_{j} = t_{j} t_{i}$ for all $i, j$
- $(t_{\sigma(1)}, t_{\sigma(2)}, \ldots, t_{\sigma(k)})$ is a path in $B(A_{n-1})$ for all $\sigma \in A_{n-1}$.

**Corollary**

$SP(A_{n-1}) \cong \text{Boolean}(k)$, the Boolean poset of rank $k$ (poset of subsets of $\{1, \ldots, k\}$ ordered by inclusion).
Example: $B_2$

$SP(B_2)$ is formed by two copies of $Boolean(2)$ that share the smallest and biggest elements.
In general, we have

**Theorem**

Let $W$ be finite Coxeter group, $w_0$ the longest element in $W$, and $\ell_0 = \ell_T(w_0)$. If $t_1 t_2 \cdots t_{\ell_0} = w_0$ then

(a) $t_i t_j = t_j t_i$ for $1 \leq i, j \leq \ell_0$. In particular $t_{\tau(1)} t_{\tau(2)} \cdots t_{\tau(\ell_0)} = w_0$ for all $\tau \in A_{\ell_0-1}$.

(b) $(t_{\tau(1)}, t_{\tau(2)}, \ldots, t_{\tau(\ell_0)})$ is a path in the Bruhat graph of $W$ for all $\tau \in A_{\ell_0-1}$

**Corollary** ($SP(W)$)

$SP(W)$ is formed by $\alpha_W$ Boolean posets of rank $\ell_0$ (that share the smallest and biggest elements).
<table>
<thead>
<tr>
<th>$W$</th>
<th>rank($SP(W)$)</th>
<th># of Boolean posets</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_n$</td>
<td>$\lfloor \frac{n}{2} \rfloor$</td>
<td>1</td>
</tr>
<tr>
<td>$B_n$</td>
<td>$n$</td>
<td>$b_n$</td>
</tr>
<tr>
<td>$D_n$</td>
<td>$n$ if $n$ is even; $n-1$ if $n$ is odd</td>
<td>$d_n$</td>
</tr>
<tr>
<td>$I_2(m)$</td>
<td>$2$ if $m$ even; $1$ if $m$ odd</td>
<td>$\frac{m}{2}$ if $m$ even; $1$ if $m$ odd</td>
</tr>
<tr>
<td>$F_4$</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$H_3$</td>
<td>3</td>
<td>5</td>
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<tr>
<td>$E_6$</td>
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<tr>
<td>$E_7$</td>
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<td>135</td>
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<tr>
<td>$E_8$</td>
<td>8</td>
<td>2025</td>
</tr>
</tbody>
</table>

\[
b_n = 1 + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{j!} \prod_{i=0}^{j-1} \left( \binom{n-2i}{2} \right)\]

\[
d_n = \frac{1}{\lfloor \frac{m}{2} \rfloor!} \prod_{i=0}^{\lfloor \frac{m}{2} \rfloor-1} \left( \binom{n-2i}{2} \right), \quad m = n \text{ if } n \text{ is even. Otherwise } m = n - 1.\]
Let $\psi(\text{Boolean}(k))$ be the $\text{cd}$-index of $\text{Boolean}(k)$ (that is, the regular $\text{cd}$-index of the Eulerian poset $\text{Boolean}(k)$). Then Ehrenborg and Readdy show that
\[
\psi(\text{Boolean}(1)) = 1
\]
\[
\psi(\text{Boolean}(k)) = \psi(\text{Boolean}(k - 1)) \cdot c + G(\psi(\text{Boolean}(k - 1))
\]
$G$ is the derivation (derivation means $G(xy) = xG(y) + G(x)y$)
$G(c) = d$ and $G(d) = \text{cd}$.
For example
\[
\psi(\text{Boolean}(2)) = c
\]
\[
\psi(\text{Boolean}(3)) = c^2 + d
\]
\[
\psi(\text{Boolean}(4)) = c^3 + 2(cd + dc)
\]

**Theorem**

*The lowest-degree terms of $\tilde{\psi}_{e,w_0}$ are given by $\alpha_W \psi(\text{Boolean}(\ell_{T}(w_0)))$ for some $\alpha_W \in \mathbb{Z}$.***
Corollary

The lowest-degree terms of $\tilde{\psi}_{e, w_0}$ are minimized (component-wise) by $\psi(\text{Boolean}(\ell_0))$.

This corollary is true for the lowest degree terms of $\psi_{e, v}$ if $[c^{\ell_0-1}] = 1$, where $[c^k]$ is denotes the coefficient of $c^k$ in $\psi_{e, v}$.

Conjecture: Corollary holds for $\tilde{\psi}_{u, v}$. 