Descent sets of cyclic permutations

Sergi Elizalde

Dartmouth College

AMS Fall Eastern Section Meeting
Special Session on Algebraic Combinatorics
Permutations

\[[n] = \{1, 2, \ldots, n\}, \quad \pi \in S_n\]
Permutations

\[ [n] = \{1, 2, \ldots, n\}, \quad \pi \in S_n \]

\[ \pi = 2517364 = (1, 2, 5, 3)(4, 7)(6) = (5, 3, 1, 2)(6)(7, 4) \]

one line notation  \hspace{1cm}  cycle notation  \hspace{1cm}  cycle notation
Permutations

\[ [n] = \{1, 2, \ldots, n\}, \quad \pi \in S_n \]

\[ \pi = 2517364 = (1, 2, 5, 3)(4, 7)(6) = (5, 3, 1, 2)(6)(7, 4) \]

one line notation \hspace{2cm} cycle notation \hspace{2cm} cycle notation

\[ C_n \subset S_n \quad \text{cyclic permutations} \]

\[ |C_n| = (n - 1)! \]

\[ C_3 = \{(1, 2, 3), (1, 3, 2)\} = \{231, 312\} \]
Permutations

\[ [n] = \{1, 2, \ldots, n\}, \quad \pi \in S_n \]

\[ \pi = \underline{2517364} = (1, 2, 5, 3)(4, 7)(6) = (5, 3, 1, 2)(6)(7, 4) \]

\[ C_n \subset S_n \quad \text{cyclic permutations} \]

\[ |C_n| = (n - 1)! \]

\[ C_3 = \{(1, 2, 3), (1, 3, 2)\} = \{231, 312\} \]

The descent set of \( \pi \in S_n \) is

\[ D(\pi) = \{i : 1 \leq i \leq n - 1, \ \pi(i) > \pi(i + 1)\}. \]

\[ D(25 \cdot 17 \cdot 36 \cdot 4) = \{2, 4, 6\} \]
Origin

Descents of cyclic permutations come up when determining the smallest number of symbols needed to realize a permutation by shifts.
Origin

Descents of cyclic permutations come up when determining the smallest number of symbols needed to realize a permutation by shifts. E.g., the permutation 4217536 can be realized using three symbols:

\[
\begin{array}{c}
2102212210 \ldots \\
102212210 \ldots \\
02212210 \ldots \\
2212210 \ldots \\
212210 \ldots \\
12210 \ldots \\
2210 \ldots \\
\end{array}
\]

\[\begin{array}{c|c}
\text{Index} & \text{Value} \\
\hline
1 & 4 \\
2 & 2 \\
3 & 1 \\
4 & 7 \\
5 & 5 \\
6 & 3 \\
7 & 6 \\
\end{array}\]

\[
\text{lexicographic order of the shifted sequences}
\]
Descents of cyclic permutations come up when determining the smallest number of symbols needed to realize a permutation by shifts. E.g., the permutation 4217536 can be realized using three symbols:

\[
\begin{array}{c}
2102212210\ldots \\
102212210\ldots \\
02212210\ldots \\
2212210\ldots \\
212210\ldots \\
12210\ldots \\
2210\ldots \\
\end{array}
\]

The number of symbols needed is related to the descents of the cycle \((4, 2, 1, 7, 5, 3, 6)\).
#### Descent sets of 5-cycles

<table>
<thead>
<tr>
<th></th>
<th>$C_5$</th>
<th></th>
<th>$C_5$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(1, 2, 3, 4, 5)</td>
<td>$2345 \cdot 1$</td>
<td>(2, 3, 1, 4, 5)</td>
<td>$4 \cdot 3 \cdot 15 \cdot 2$</td>
<td></td>
</tr>
<tr>
<td>(2, 1, 3, 4, 5)</td>
<td>$3 \cdot 145 \cdot 2$</td>
<td>(2, 4, 3, 1, 5)</td>
<td>$5 \cdot 4 \cdot 13 \cdot 2$</td>
<td></td>
</tr>
<tr>
<td>(3, 2, 1, 4, 5)</td>
<td>$4 \cdot 125 \cdot 3$</td>
<td>(4, 2, 3, 1, 5)</td>
<td>$5 \cdot 3 \cdot 124$</td>
<td></td>
</tr>
<tr>
<td>(4, 3, 2, 1, 5)</td>
<td>$5 \cdot 1234$</td>
<td>(1, 4, 2, 3, 5)</td>
<td>$4 \cdot 35 \cdot 2 \cdot 1$</td>
<td></td>
</tr>
<tr>
<td>(1, 3, 2, 4, 5)</td>
<td>$34 \cdot 25 \cdot 1$</td>
<td>(2, 1, 4, 3, 5)</td>
<td>$4 \cdot 15 \cdot 3 \cdot 2$</td>
<td></td>
</tr>
<tr>
<td>(1, 4, 3, 2, 5)</td>
<td>$45 \cdot 23 \cdot 1$</td>
<td>(2, 3, 4, 1, 5)</td>
<td>$5 \cdot 34 \cdot 12$</td>
<td></td>
</tr>
<tr>
<td>(3, 1, 2, 4, 5)</td>
<td>$24 \cdot 15 \cdot 3$</td>
<td>(3, 4, 2, 1, 5)</td>
<td>$5 \cdot 14 \cdot 23$</td>
<td></td>
</tr>
<tr>
<td>(3, 1, 4, 2, 5)</td>
<td>$45 \cdot 123$</td>
<td>(4, 2, 1, 3, 5)</td>
<td>$3 \cdot 15 \cdot 24$</td>
<td></td>
</tr>
<tr>
<td>(4, 3, 1, 2, 5)</td>
<td>$25 \cdot 134$</td>
<td>(1, 3, 4, 2, 5)</td>
<td>$35 \cdot 4 \cdot 2 \cdot 1$</td>
<td></td>
</tr>
<tr>
<td>(1, 2, 4, 3, 5)</td>
<td>$245 \cdot 3 \cdot 1$</td>
<td>(3, 4, 1, 2, 5)</td>
<td>$25 \cdot 4 \cdot 13$</td>
<td></td>
</tr>
<tr>
<td>(2, 4, 1, 3, 5)</td>
<td>$345 \cdot 12$</td>
<td>(4, 1, 3, 2, 5)</td>
<td>$35 \cdot 2 \cdot 14$</td>
<td></td>
</tr>
<tr>
<td>(4, 1, 2, 3, 5)</td>
<td>$235 \cdot 14$</td>
<td>(3, 2, 4, 1, 5)</td>
<td>$5 \cdot 4 \cdot 2 \cdot 13$</td>
<td></td>
</tr>
</tbody>
</table>
### Descent sets of 5-cycles

<table>
<thead>
<tr>
<th>$C_5$</th>
<th>$S_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(1, 2, 3, 4, 5) = 2345 \cdot 1$</td>
<td>1234</td>
</tr>
<tr>
<td>$(2, 1, 3, 4, 5) = 3 \cdot 145 \cdot 2$</td>
<td>2 \cdot 134</td>
</tr>
<tr>
<td>$(3, 2, 1, 4, 5) = 4 \cdot 125 \cdot 3$</td>
<td>3 \cdot 124</td>
</tr>
<tr>
<td>$(4, 3, 2, 1, 5) = 5 \cdot 1234$</td>
<td>4 \cdot 123</td>
</tr>
<tr>
<td>$(1, 3, 2, 4, 5) = 34 \cdot 25 \cdot 1$</td>
<td>13 \cdot 24</td>
</tr>
<tr>
<td>$(1, 4, 3, 2, 5) = 45 \cdot 23 \cdot 1$</td>
<td>14 \cdot 23</td>
</tr>
<tr>
<td>$(3, 1, 2, 4, 5) = 24 \cdot 15 \cdot 3$</td>
<td>23 \cdot 14</td>
</tr>
<tr>
<td>$(3, 1, 4, 2, 5) = 45 \cdot 123$</td>
<td>34 \cdot 12</td>
</tr>
<tr>
<td>$(4, 3, 1, 2, 5) = 25 \cdot 134$</td>
<td>24 \cdot 13</td>
</tr>
<tr>
<td>$(1, 2, 4, 3, 5) = 245 \cdot 3 \cdot 1$</td>
<td>124 \cdot 3</td>
</tr>
<tr>
<td>$(2, 4, 1, 3, 5) = 345 \cdot 12$</td>
<td>134 \cdot 2</td>
</tr>
<tr>
<td>$(4, 1, 2, 3, 5) = 235 \cdot 14$</td>
<td>234 \cdot 1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$C_5$</th>
<th>$S_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(2, 3, 1, 4, 5) = 4 \cdot 3 \cdot 15 \cdot 2$</td>
<td>3 \cdot 2 \cdot 14</td>
</tr>
<tr>
<td>$(2, 4, 3, 1, 5) = 5 \cdot 4 \cdot 13 \cdot 2$</td>
<td>4 \cdot 2 \cdot 13</td>
</tr>
<tr>
<td>$(4, 2, 3, 1, 5) = 5 \cdot 3 \cdot 124$</td>
<td>4 \cdot 3 \cdot 12</td>
</tr>
<tr>
<td>$(1, 4, 2, 3, 5) = 4 \cdot 35 \cdot 2 \cdot 1$</td>
<td>3 \cdot 24 \cdot 1</td>
</tr>
<tr>
<td>$(2, 1, 4, 3, 5) = 4 \cdot 15 \cdot 3 \cdot 2$</td>
<td>2 \cdot 14 \cdot 3</td>
</tr>
<tr>
<td>$(2, 3, 4, 1, 5) = 5 \cdot 34 \cdot 12$</td>
<td>4 \cdot 23 \cdot 1</td>
</tr>
<tr>
<td>$(3, 4, 2, 1, 5) = 5 \cdot 14 \cdot 23$</td>
<td>4 \cdot 13 \cdot 2</td>
</tr>
<tr>
<td>$(4, 2, 1, 3, 5) = 3 \cdot 15 \cdot 24$</td>
<td>3 \cdot 14 \cdot 2</td>
</tr>
<tr>
<td>$(1, 3, 4, 2, 5) = 35 \cdot 4 \cdot 2 \cdot 1$</td>
<td>14 \cdot 3 \cdot 2</td>
</tr>
<tr>
<td>$(3, 4, 1, 2, 5) = 25 \cdot 4 \cdot 13$</td>
<td>24 \cdot 3 \cdot 1</td>
</tr>
<tr>
<td>$(4, 1, 3, 2, 5) = 35 \cdot 2 \cdot 14$</td>
<td>34 \cdot 2 \cdot 1</td>
</tr>
<tr>
<td>$(3, 2, 4, 1, 5) = 5 \cdot 4 \cdot 2 \cdot 13$</td>
<td>4 \cdot 3 \cdot 2 \cdot 1</td>
</tr>
</tbody>
</table>

Sergi Elizalde

Descent sets of cyclic permutations
Theorem

For every $n$ there is a bijection $\varphi : C_{n+1} \rightarrow S_n$ such that if $\pi \in C_{n+1}$ and $\sigma = \varphi(\pi)$, then

$$D(\pi) \cap [n-1] = D(\sigma).$$
The bijection $\varphi : \mathcal{C}_{n+1} \rightarrow S_n$. First step

Given $\pi \in \mathcal{C}_{n+1}$, write it in cycle form with $n + 1$ at the end:

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21) \in \mathcal{C}_{21}$$
The bijection $\varphi : \mathcal{C}_{n+1} \rightarrow S_n$. First step

Given $\pi \in \mathcal{C}_{n+1}$, write it in cycle form with $n + 1$ at the end:

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21) \in C_{21}.$$

Delete $n + 1$ and split at the “left-to-right maxima”:

$$\sigma = (11, 4, 10, 1, 7)(16, 9, 3, 5, 12)(20, 2, 6, 14, 18, 8, 13, 19, 15, 17) \in S_{20}.$$
The bijection $\varphi : C_{n+1} \rightarrow S_n$. First step

Given $\pi \in C_{n+1}$, write it in cycle form with $n + 1$ at the end:

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21) \in C_{21}$$

Delete $n + 1$ and split at the “left-to-right maxima”:

$$\sigma = (11, 4, 10, 1, 7)(16, 9, 3, 5, 12)(20, 2, 6, 14, 18, 8, 13, 19, 15, 17) \in S_{20}.$$ 

This map $\pi \mapsto \sigma$ is a bijection.
The bijection $\varphi : \mathcal{C}_{n+1} \rightarrow \mathcal{S}_n$. \textit{First step}

Given $\pi \in \mathcal{C}_{n+1}$, write it in cycle form with $n+1$ at the end:

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21) \in \mathcal{C}_{21}$$

Delete $n+1$ and split at the “left-to-right maxima”:

$$\sigma = (11, 4, 10, 1, 7)(16, 9, 3, 5, 12)(20, 2, 6, 14, 18, 8, 13, 19, 15, 17) \in \mathcal{S}_{20}.$$

This map $\pi \mapsto \sigma$ is a bijection, but unfortunately it does not always preserve the descent set:

$$\pi(7) = 16 > \pi(8) = 13 \quad \text{but} \quad \sigma(7) = 11 < \sigma(8) = 13.$$
The bijection $\varphi : C_{n+1} \rightarrow S_n$. First step

Given $\pi \in C_{n+1}$, write it in cycle form with $n + 1$ at the end:

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21) \in C_{21}.$$

Delete $n + 1$ and split at the “left-to-right maxima”:

$$\sigma = (11, 4, 10, 1, 7)(16, 9, 3, 5, 12)(20, 2, 6, 14, 18, 8, 13, 19, 15, 17) \in S_{20}.$$

This map $\pi \mapsto \sigma$ is a bijection, but unfortunately it does not always preserve the descent set:

$$\pi(7) = 16 > \pi(8) = 13 \quad \text{but} \quad \sigma(7) = 11 < \sigma(8) = 13.$$

We say that the pair $\{7, 8\}$ is bad. We will fix the bad pairs.
The bijection $\varphi : \mathcal{C}_{n+1} \to S_n$. Fixing bad pairs

$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$

$\sigma = (11, 4, 10, 1, 7)(16, 9, 3, 5, 12)(20, 2, 6, 14, 18, 8, 13, 19, 15, 17)$
The bijection $\varphi: C_{n+1} \rightarrow S_n$. Fixing bad pairs

For each but the last cycle of $\sigma$, from left to right:

\[
\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)
\]
\[
\sigma = (11, 4, 10, 1, 7)(16, 9, 3, 5, 12)(20, 2, 6, 14, 18, 8, 13, 19, 15, 17)
\]
The bijection $\varphi : \mathcal{C}_{n+1} \rightarrow S_n$. Fixing bad pairs

For each but the last cycle of $\sigma$, from left to right:

$\triangleright z :=$ rightmost entry of the cycle.

\[
\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21) \\
\sigma = (11, 4, 10, 1, 7)(16, 9, 3, 5, 12)(20, 2, 6, 14, 18, 8, 13, 19, 15, 17) \\
z := 7.
\]
The bijection $\varphi : \mathcal{C}_{n+1} \rightarrow S_n$. Fixing bad pairs

For each but the last cycle of $\sigma$, from left to right:

1. $z :=$ rightmost entry of the cycle.
   - If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.

\[ \pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21) \]
\[ \sigma = (11, 4, 10, 1, 7)(16, 9, 3, 5, 12)(20, 2, 6, 14, 18, 8, 13, 19, 15, 17) \]
\[ \{7, 6\} \text{ and } \{7, 8\} \text{ are bad; and } \sigma(6) = 14 > 13 = \sigma(8) \Rightarrow \varepsilon := -1. \]
The bijection $\varphi : C_{n+1} \rightarrow S_n$. Fixing bad pairs

For each but the last cycle of $\sigma$, from left to right:

- $z :=$ rightmost entry of the cycle.
  - If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.

- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch $z$ and $z+\varepsilon$ (in the cycle form of $\sigma$).

\[\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)\]
\[\sigma = (11, 4, 10, 1, 7)(16, 9, 3, 5, 12)(20, 2, 6, 14, 18, 8, 13, 19, 15, 17)\]
\[z := 7.\]
Switch 7 and 6.
The bijection \( \varphi : \mathcal{C}_{n+1} \to S_n \). Fixing bad pairs

For each but the last cycle of \( \sigma \), from left to right:

- \( z := \) rightmost entry of the cycle.
  
  If \( \{z, z-1\} \) or \( \{z, z+1\} \) are bad, let \( \varepsilon = \pm 1 \) be such that
  \( \{z, z+\varepsilon\} \) is bad and \( \sigma(z+\varepsilon) \) is largest.

- Repeat for as long as \( \{z, z+\varepsilon\} \) is bad:
  1. Switch \( z \) and \( z+\varepsilon \) (in the cycle form of \( \sigma \)).

\[
\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)
\]
\[
\sigma = (11, 4, 10, 1, 6)(16, 9, 3, 5, 12)(20, 2, 7, 14, 18, 8, 13, 19, 15, 17)
\]
\[
z := 7.
\]
\[
\varepsilon := -1.
\]

Switch 7 and 6.
The bijection $\varphi : C_{n+1} \rightarrow S_n$. Fixing bad pairs

For each but the last cycle of $\sigma$, from left to right:

1. $z :=$ rightmost entry of the cycle.
   - If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.

2. Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
   - Switch $z$ and $z+\varepsilon$ (in the cycle form of $\sigma$).
   - If the elements preceding the last switched entries have consecutive values, switch them.

\begin{align*}
\pi &= (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21) \\
\sigma &= (11, 4, 10, 1, 6)(16, 9, 3, 5, 12)(20, 2, 7, 14, 18, 8, 13, 19, 15, 17) \\
z &:= 7. \\
\varepsilon &:= -1.
\end{align*}

Switch 7 and 6. Switch 1 and 2.
The bijection $\varphi : C_{n+1} \to S_n$. Fixing bad pairs

For each but the last cycle of $\sigma$, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.

- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch $z$ and $z+\varepsilon$ (in the cycle form of $\sigma$).
  2. If the elements preceding the last switched entries have consecutive values, switch them.

$$
\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)
$$
$$
\sigma = (11, 4, 10, 2, 6)(16, 9, 3, 5, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)
$$
$$
z := 7.
$$
$$
\varepsilon := -1.
$$
Switch 7 and 6. Switch 1 and 2.
The bijection $\varphi : C_{n+1} \rightarrow S_n$. Fixing bad pairs

For each but the last cycle of $\sigma$, from left to right:

- $z$ := rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.

- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch $z$ and $z+\varepsilon$ (in the cycle form of $\sigma$).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, \underline{10}, 2, 6)(16, 9, 3, 5, 12)(\underline{20}, 1, 7, 14, 18, 8, 13, 19, 15, 17)$$
$$z := 7.$$  
$$\varepsilon := -1.$$  
Switch 7 and 6. Switch 1 and 2.
The bijection $\varphi : C_{n+1} \rightarrow S_n$. Fixing bad pairs

For each but the last cycle of $\sigma$, from left to right:

1. $z :=$ rightmost entry of the cycle.

2. If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.

3. Repeat for as long as $\{z, z+\varepsilon\}$ is bad.

Repeat for as long as $\{z, z+\varepsilon\}$ is bad.

Switch $z$ and $z+\varepsilon$ (in the cycle form of $\sigma$) if the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.

$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 1)$

$\sigma = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$

$z := 6, \varepsilon := -1$.
The bijection \( \varphi : \mathcal{C}_{n+1} \to \mathcal{S}_n \). Fixing bad pairs

For each but the last cycle of \( \sigma \), from left to right:

- \( z := \) rightmost entry of the cycle. If \( \{z, z-1\} \) or \( \{z, z+1\} \) are bad, let \( \varepsilon = \pm 1 \) be such that \( \{z, z+\varepsilon\} \) is bad and \( \sigma(z+\varepsilon) \) is largest.

- Repeat for as long as \( \{z, z+\varepsilon\} \) is bad:
  1. Switch \( z \) and \( z+\varepsilon \) (in the cycle form of \( \sigma \)).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. \( z := \) new rightmost entry of the cycle.

\[
\begin{align*}
\pi &= (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21) \\
\sigma &= (11, 4, 10, 2, 6)(16, 9, 3, 5, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17) \\
z &:= 6. \quad \{6, 5\} \text{ is bad.} \quad \varepsilon := -1.
\end{align*}
\]
The bijection $\varphi : C_{n+1} \to S_n$. Fixing bad pairs

For each but the last cycle of $\sigma$, from left to right:

- $z :=$ rightmost entry of the cycle.
  
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.

- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  
  1. Switch $z$ and $z+\varepsilon$ (in the cycle form of $\sigma$).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$

$$\sigma = (11, 4, 10, 2, 6)(16, 9, 3, 5, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$$

$z := 6.$

Switch 6 and 5. $\varepsilon := -1.$
The bijection $\varphi : C_{n+1} \rightarrow S_n$. Fixing bad pairs

For each but the last cycle of $\sigma$, from left to right:

- $z := \text{rightmost entry of the cycle.}$
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.

- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch $z$ and $z+\varepsilon$ (in the cycle form of $\sigma$).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z := \text{new rightmost entry of the cycle.}$

$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$

$\sigma = (11, 4, 10, 2, 5)(16, 9, 3, 6, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$

$z := 6.$

Switch 6 and 5.

$\varepsilon := -1.$
The bijection $\varphi : C_{n+1} \rightarrow S_n$. Fixing bad pairs

For each but the last cycle of $\sigma$, from left to right:

1. $z :=$ rightmost entry of the cycle.
   
   If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.

2. Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
   1. Switch $z$ and $z+\varepsilon$ (in the cycle form of $\sigma$).
   2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
   3. $z :=$ new rightmost entry of the cycle.

\[\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)\]
\[\sigma = (11, 4, 10, 2, 5)(16, 9, 3, 6, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)\]
\[z := 6.\]
\[\varepsilon := -1.\]
Switch 6 and 5. Switch 2 and 3.
The bijection $\varphi : \mathcal{C}_{n+1} \rightarrow S_n$. Fixing bad pairs

For each but the last cycle of $\sigma$, from left to right:

1. $z :=$ rightmost entry of the cycle.
   If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.

2. Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
   1. Switch $z$ and $z+\varepsilon$ (in the cycle form of $\sigma$).
   2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
   3. $z :=$ new rightmost entry of the cycle.

$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$
$\sigma = (11, 4, 10, 3, 5)(16, 9, 2, 6, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$
$z := 6.$
Switch 6 and 5. Switch 2 and 3.
$\varepsilon := -1.$
The bijection $\varphi : C_{n+1} \rightarrow S_n$. Fixing bad pairs

For each but the last cycle of $\sigma$, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.

- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch $z$ and $z+\varepsilon$ (in the cycle form of $\sigma$).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 10, 3, 5)(16, 9, 2, 6, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$$

$z := 6$. $\varepsilon := -1$.

Switch 6 and 5. Switch 2 and 3. Switch 10 and 9.
The bijection $\varphi : C_{n+1} \to S_n$. Fixing bad pairs

For each but the last cycle of $\sigma$, from left to right:

1. $z := \text{rightmost entry of the cycle.}$
   If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.

2. Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
   1. Switch $z$ and $z+\varepsilon$ (in the cycle form of $\sigma$).
   2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
   3. $z := \text{new rightmost entry of the cycle.}$

Example:

$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$

$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$

$z := 6.$

Switch 6 and 5. Switch 2 and 3. Switch 10 and 9.
The bijection $\varphi : \mathcal{C}_{n+1} \rightarrow S_n$. Fixing bad pairs

For each but the last cycle of $\sigma$, from left to right:

- $z :=$ rightmost entry of the cycle.
  - If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.

- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch $z$ and $z+\varepsilon$ (in the cycle form of $\sigma$).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$  
$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$  
$z := 5$.  
$\varepsilon := -1$.  

Sergi Elizalde  
Descent sets of cyclic permutations
The bijection $\varphi : \mathcal{C}_{n+1} \rightarrow S_n$. Fixing bad pairs

For each but the last cycle of $\sigma$, from left to right:

- $z :=$ rightmost entry of the cycle.
  
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.

- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch $z$ and $z+\varepsilon$ (in the cycle form of $\sigma$).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$

$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$

$z := 5$. $\{5, 4\}$ is OK, so we move on to the second cycle.
The bijection $\varphi : C_{n+1} \to S_n$. Fixing bad pairs

For each but the last cycle of $\sigma$, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.

- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch $z$ and $z+\varepsilon$ (in the cycle form of $\sigma$).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$$
$$z := 12.$$
The bijection $\varphi : C_{n+1} \rightarrow S_n$. Fixing bad pairs

For each but the last cycle of $\sigma$, from left to right:

- $z :=$ rightmost entry of the cycle.
  
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.

- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch $z$ and $z+\varepsilon$ (in the cycle form of $\sigma$).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$

$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$

$z := 12$. $\{12, 11\}$ is OK but $\{12, 13\}$ is bad $\Rightarrow \varepsilon := 1$. 
The bijection $\varphi : C_{n+1} \rightarrow S_n$. Fixing bad pairs

For each but the last cycle of $\sigma$, from left to right:

- $z :=$ rightmost entry of the cycle.
  
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.

- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  
  1. Switch $z$ and $z+\varepsilon$ (in the cycle form of $\sigma$).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 12)(20, 1, 7, 14, 18, 8, 13, 19, 15, 17)$$


$\varepsilon := 1.$
The bijection $\varphi : \mathcal{C}_{n+1} \rightarrow S_n$. Fixing bad pairs

For each but the last cycle of $\sigma$, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that
  $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.

- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch $z$ and $z+\varepsilon$ (in the cycle form of $\sigma$).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$
$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 13)(20, 1, 7, 14, 18, 8, 12, 19, 15, 17)$
$z := 12.$
Switch 12 and 13.

$\varepsilon := 1.$
The bijection $\varphi : \mathcal{C}_{n+1} \rightarrow S_n$. Fixing bad pairs

For each but the last cycle of $\sigma$, from left to right:

1. $z := \text{rightmost entry of the cycle}$. If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.

2. Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
   1. Switch $z$ and $z+\varepsilon$ (in the cycle form of $\sigma$).
   2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
   3. $z := \text{new rightmost entry of the cycle}$.

$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$

$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 13)(20, 1, 7, 14, 18, 8, 12, 19, 15, 17)$

The bijection $\varphi : C_{n+1} \rightarrow S_n$. Fixing bad pairs

For each but the last cycle of $\sigma$, from left to right:

- $z :=$ rightmost entry of the cycle.
  
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.

- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch $z$ and $z+\varepsilon$ (in the cycle form of $\sigma$).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$

$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 13)(20, 1, 7, 14, 18, 8, 12, 19, 15, 17)$$

$z := 13.$

$\varepsilon := 1.$
The bijection $\varphi : C_{n+1} \rightarrow S_n$. Fixing bad pairs

For each but the last cycle of $\sigma$, from left to right:

- $z :=$ rightmost entry of the cycle. If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.

- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch $z$ and $z+\varepsilon$ (in the cycle form of $\sigma$).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 13)(20, 1, 7, 14, 18, 8, 12, 19, 15, 17)$$
$$z := 13. \quad \{13, 14\} \text{ is bad.} \quad \varepsilon := 1.$$
The bijection $\varphi : C_{n+1} \to S_n$. Fixing bad pairs

For each but the last cycle of $\sigma$, from left to right:

1. $z :=$ rightmost entry of the cycle.
   If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.

2. Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
   1. Switch $z$ and $z+\varepsilon$ (in the cycle form of $\sigma$).
   2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
   3. $z :=$ new rightmost entry of the cycle.

$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$

$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 13)(20, 1, 7, 14, 18, 8, 12, 19, 15, 17)$

$z := 13.
\varepsilon := 1.$

Switch 13 and 14.
The bijection $\varphi : \mathcal{C}_{n+1} \rightarrow S_n$. Fixing bad pairs

For each but the last cycle of $\sigma$, from left to right:

- $z :=$ rightmost entry of the cycle.
  - If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.

- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch $z$ and $z+\varepsilon$ (in the cycle form of $\sigma$).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$

$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 14)(20, 1, 7, 13, 18, 8, 12, 19, 15, 17)$

$z := 13.$

Switch 13 and 14.

$\varepsilon := 1.$
The bijection $\varphi : C_{n+1} \rightarrow S_n$. Fixing bad pairs

For each but the last cycle of $\sigma$, from left to right:

- $z :=$ rightmost entry of the cycle.
  - If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.

- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch $z$ and $z+\varepsilon$ (in the cycle form of $\sigma$).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$

$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 6, 14)(20, 1, 7, 13, 18, 8, 12, 19, 15, 17)$

$z := 13$.

Switch 13 and 14. Switch 6 and 7.

$\varepsilon := 1$. 
The bijection $\varphi : C_{n+1} \to S_n$. Fixing bad pairs

For each but the last cycle of $\sigma$, from left to right:

1. $z :=$ rightmost entry of the cycle.
   
   If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.

2. Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
   
   1. Switch $z$ and $z+\varepsilon$ (in the cycle form of $\sigma$).
   2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
   3. $z :=$ new rightmost entry of the cycle.

$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$

$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 7, 14)(20, 1, 6, 13, 18, 8, 12, 19, 15, 17)$

$z := 13.$

Switch 13 and 14. Switch 6 and 7.
The bijection $\varphi : C_{n+1} \rightarrow S_n$. Fixing bad pairs

For each but the last cycle of $\sigma$, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.

- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch $z$ and $z+\varepsilon$ (in the cycle form of $\sigma$).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 2, 7, 14)(20, 1, 6, 13, 18, 8, 12, 19, 15, 17)$$
$$z := 13.$$  \quad \varepsilon := 1.$$
The bijection \( \varphi : C_{n+1} \rightarrow S_n \). Fixing bad pairs

For each but the last cycle of \( \sigma \), from left to right:

- \( z := \) rightmost entry of the cycle.
  
  If \( \{z, z-1\} \) or \( \{z, z+1\} \) are bad, let \( \varepsilon = \pm 1 \) be such that \( \{z, z+\varepsilon\} \) is bad and \( \sigma(z+\varepsilon) \) is largest.

- Repeat for as long as \( \{z, z+\varepsilon\} \) is bad:
  1. Switch \( z \) and \( z+\varepsilon \) (in the cycle form of \( \sigma \)).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. \( z := \) new rightmost entry of the cycle.

\[
\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)
\]
\[
\sigma = (11, 4, 9, 3, 5)(16, 10, 1, 7, 14)(20, 2, 6, 13, 18, 8, 12, 19, 15, 17)
\]
\[
z := 13.
\]
\[
\varepsilon := 1.
\]
The bijection $\varphi : \mathcal{C}_{n+1} \to S_n$. Fixing bad pairs

For each but the last cycle of $\sigma$, from left to right:

- $z :=$ rightmost entry of the cycle.
  
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.

- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  
  1. Switch $z$ and $z+\varepsilon$ (in the cycle form of $\sigma$).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$

$\sigma = (11, 4, 9, 3, 5)(16, 10, 1, 7, 14)(20, 2, 6, 13, 18, 8, 12, 19, 15, 17)$

$z := 14$.

$\varepsilon := 1$. 
The bijection $\varphi : \mathcal{C}_{n+1} \rightarrow S_n$. Fixing bad pairs

For each but the last cycle of $\sigma$, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.

- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch $z$ and $z+\varepsilon$ (in the cycle form of $\sigma$).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)
\sigma = (11, 4, 9, 3, 5)(16, 10, 1, 7, 14)(20, 2, 6, 13, 18, 8, 12, 19, 15, 17)

$z := 14$. $\{14, 15\}$ is bad. $\varepsilon := 1$. 
The bijection $\varphi : C_{n+1} \rightarrow S_n$. Fixing bad pairs

For each but the last cycle of $\sigma$, from left to right:

1. $z :=$ rightmost entry of the cycle.
   If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.

2. Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
   1. Switch $z$ and $z+\varepsilon$ (in the cycle form of $\sigma$).
   2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
   3. $z :=$ new rightmost entry of the cycle.

$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$

$\sigma = (11, 4, 9, 3, 5)(16, 10, 1, 7, 14)(20, 2, 6, 13, 18, 8, 12, 19, 15, 17)$

$z := 14.$

Switch 14 and 15.
The bijection $\varphi : C_{n+1} \to S_n$. Fixing bad pairs

For each but the last cycle of $\sigma$, from left to right:

1. $z :=$ rightmost entry of the cycle.
   
   If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.

2. Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
   
   1. Switch $z$ and $z+\varepsilon$ (in the cycle form of $\sigma$).
   2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
   3. $z :=$ new rightmost entry of the cycle.

$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$

$\sigma = (11, 4, 9, 3, 5)(16, 10, 1, 7, 15)(20, 2, 6, 13, 18, 8, 12, 19, 14, 17)$

$z := 14$.

Switch 14 and 15.
The bijection $\varphi : \mathcal{C}_{n+1} \rightarrow S_n$. Fixing bad pairs

For each but the last cycle of $\sigma$, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.

- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch $z$ and $z+\varepsilon$ (in the cycle form of $\sigma$).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$
\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21) \\
\sigma = (11, 4, 9, 3, 5)(16, 10, 1, 7, 15)(20, 2, 6, 13, 18, 8, 12, 19, 14, 17) \\
z := 14. \\
\varepsilon := 1.
$$

Switch 14 and 15.
The bijection $\varphi : C_{n+1} \to S_n$. Fixing bad pairs

For each but the last cycle of $\sigma$, from left to right:

- $z :=$ rightmost entry of the cycle.
  If $\{z, z+1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.

- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch $z$ and $z+\varepsilon$ (in the cycle form of $\sigma$).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$

$\sigma = (11, 4, 9, 3, 5)(16, 10, 1, 7, 15)(20, 2, 6, 13, 18, 8, 12, 19, 14, 17)$

$z := 15.$

$\varepsilon := 1.$
The bijection $\varphi : C_{n+1} \rightarrow S_n$. Fixing bad pairs

For each but the last cycle of $\sigma$, from left to right:

- $z :=$ rightmost entry of the cycle.
  - If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.

- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch $z$ and $z+\varepsilon$ (in the cycle form of $\sigma$).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z :=$ new rightmost entry of the cycle.

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$
$$\sigma = (11, 4, 9, 3, 5)(16, 10, 1, 7, 15)(20, 2, 6, 13, 18, 8, 12, 19, 14, 17)$$
$$z := 15. \quad \{15, 16\} \text{ is OK, so we are done.}$$
The bijection $\varphi : \mathcal{C}_{n+1} \rightarrow S_n$. Fixing bad pairs

For each but the last cycle of $\sigma$, from left to right:

- $z := $ rightmost entry of the cycle.
  - If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is largest.

- Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
  1. Switch $z$ and $z+\varepsilon$ (in the cycle form of $\sigma$).
  2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
  3. $z := $ new rightmost entry of the cycle.

$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$
$\varphi(\pi) = (11, 4, 9, 3, 5)(16, 10, 1, 7, 15)(20, 2, 6, 13, 18, 8, 12, 19, 14, 17)$

Define $\varphi(\pi) = \sigma$. 
The descent sets are preserved

$$\pi = (11, 4, 10, 1, 7, 16, 9, 3, 5, 12, 20, 2, 6, 14, 18, 8, 13, 19, 15, 17, 21)$$

$$\varphi(\pi) = (11, 4, 9, 3, 5)(16, 10, 1, 7, 15)(20, 2, 6, 13, 18, 8, 12, 19, 14, 17)$$

In one-line notation,

$$\pi = 7 \cdot 6 \cdot 5 \ 10 \ 12 \ 14 \ 16 \ 13 \ 3 \ 1 \ 4 \ 20 \ 19 \ 18 \ 16 \ 9 \ 21 \ 8 \ 15 \ 2 \ 11$$

$$\varphi(\pi) = 7 \cdot 6 \cdot 5 \ 9 \ 11 \ 13 \ 15 \ 12 \ 3 \ 1 \ 4 \ 19 \ 18 \ 17 \ 16 \ 10 \ 20 \ 8 \ 14 \ 2$$
The inverse map $\varphi^{-1} : S_n \rightarrow C_{n+1}$. First step

Given $\sigma \in S_n$, write it in cycle form with the largest element of each cycle first, ordering the cycles by increasing first element:

$$\sigma = (11, 4, 9, 3, 5)(16, 10, 1, 7, 15)(20, 2, 6, 13, 18, 8, 12, 19, 14, 17) \in S_{20}.$$
The inverse map $\varphi^{-1} : S_n \to C_{n+1}$. First step

Given $\sigma \in S_n$, write it in cycle form with the largest element of each cycle first, ordering the cycles by increasing first element:

$$\sigma = (11, 4, 9, 3, 5)(16, 10, 1, 7, 15)(20, 2, 6, 13, 18, 8, 12, 19, 14, 17) \in S_{20}.$$

Remove parentheses and append $n + 1$:

$$\pi = (11, 4, 9, 3, 5, 16, 10, 1, 7, 15, 20, 2, 6, 13, 18, 8, 12, 19, 14, 17, 21) \in C_{21}.$$
The inverse map $\varphi^{-1} : S_n \to C_{n+1}$. First step

Given $\sigma \in S_n$, write it in cycle form with the largest element of each cycle first, ordering the cycles by increasing first element:

$$\sigma = (11, 4, 9, 3, 5)(16, 10, 1, 7, 15)(20, 2, 6, 13, 18, 8, 12, 19, 14, 17) \in S_{20}.$$

Remove parentheses and append $n + 1$:

$$\pi = (11, 4, 9, 3, 5, 16, 10, 1, 7, 15, 20, 2, 6, 13, 18, 8, 12, 19, 14, 17, 21) \in C_{21}.$$

A pair $\{i, i + 1\}$ is bad if $\pi(i) > \pi(i + 1)$ but $\sigma(i) < \sigma(i + 1)$, or viceversa. We will fix the bad pairs.
The inverse map $\varphi^{-1} : S_n \rightarrow C_{n+1}$. First step

Given $\sigma \in S_n$, write it in cycle form with the largest element of each cycle first, ordering the cycles by increasing first element:

$$\sigma = (11, 4, 9, 3, 5)(16, 10, 1, 7, 15)(20, 2, 6, 13, 18, 8, 12, 19, 14, 17) \in S_{20}.$$

Remove parentheses and append $n + 1$:

$$\pi = (11, 4, 9, 3, 5, 16, 10, 1, 7, 15, 20, 2, 6, 13, 18, 8, 12, 19, 14, 17, 21) \in C_{21}.$$

A pair $\{i, i + 1\}$ is bad if $\pi(i) > \pi(i + 1)$ but $\sigma(i) < \sigma(i + 1)$, or viceversa. We will fix the bad pairs.

Call blocks the pieces of $\pi$ between removed parentheses.
The inverse map $\varphi^{-1}: S_n \to C_{n+1}$. Fixing bad pairs

For each but the last block of $\pi$, from right to left:

1. $z :=$ rightmost entry of the block.
   If $\{z, z-1\}$ or $\{z, z+1\}$ are bad, let $\varepsilon = \pm 1$ be such that $\{z, z+\varepsilon\}$ is bad and $\sigma(z+\varepsilon)$ is smallest.

2. Repeat for as long as $\{z, z+\varepsilon\}$ is bad:
   1. Switch $z$ and $z+\varepsilon$ (in the cycle form of $\pi$).
   2. If the elements preceding the last switched entries have consecutive values, switch them. Repeat 2.
   3. $z :=$ new rightmost entry of the block.

We obtain $\pi = \varphi^{-1}(\sigma)$.
Necklaces

\[ X = \{x_1, x_2, \ldots \} < \text{linearly ordered alphabet.} \]

A *necklace* of length \( \ell \) is a circular arrangement of \( \ell \) beads labeled with elements of \( X \), up to cyclic rotation.
Necklaces

\[ X = \{x_1, x_2, \ldots \} \]  linearly ordered alphabet.

A necklace of length \( \ell \) is a circular arrangement of \( \ell \) beads labeled with elements of \( X \), up to cyclic rotation.

Given a multiset of necklaces,

- its type is the partition whose parts are the lengths of the necklaces;

Sergi Elizalde
Descent sets of cyclic permutations
Necklaces

\[ X = \{x_1, x_2, \ldots \} \quad \text{linearly ordered alphabet.} \]

A necklace of length \( \ell \) is a circular arrangement of \( \ell \) beads labeled with elements of \( X \), up to cyclic rotation.

Given a multiset of necklaces,

- its type is the partition whose parts are the lengths of the necklaces;
- its evaluation is the monomial \( x_1^{e_1} x_2^{e_2} \ldots \) where \( e_i \) is the number of beads with label \( x_i \).
Theorem (Gessel, Reutenauer ’93)

$$\left| \{ \pi \in S_n \text{ with cycle structure } \lambda \text{ and descent composition } C \} \right| = \langle S_C, L_\lambda \rangle,$$
Permutations and necklaces

Theorem (Gessel, Reutenauer '93)

\[ |\{ \pi \in S_n \text{ with cycle structure } \lambda \text{ and descent composition } C \}| = \langle S_C, L_\lambda \rangle, \]

where

\[ S_C = \text{skew Schur function corresponding to } C, \]
\[ L_\lambda = \sum_M \text{ev}(M) \text{ over multisets } M \text{ of necklaces of type } \lambda. \]
Permutations and necklaces

Theorem (Gessel, Reutenauer ’93)

\[ |\{ \pi \in S_n \text{ with cycle structure } \lambda \text{ and descent composition } C \}| = \langle S_C, L_\lambda \rangle, \]

where

\[ S_C = \text{skew Schur function corresponding to } C, \]

\[ L_\lambda = \sum_M \text{ev}(M) \quad \text{over multisets } M \text{ of necklaces of type } \lambda. \]

Corollary (Gessel, Reutenauer ’93)

Let \( I = \{i_1, i_2, \ldots, i_k\} < \subseteq [n-1], \quad \lambda \vdash n. \) Then

\[ |\{ \pi \in S_n \text{ with cycle structure } \lambda \text{ and } D(\pi) \subseteq I \}| = |\{ \text{multisets of necklaces of type } \lambda \text{ and evaluation } x_1^{i_1}x_2^{i_2-i_1} \ldots x_k^{i_k-i_{k-1}}x_{n-i_k} \}|. \]
Non-bijective proof using Gessel-Reutenauer

Goal: $\left| \{ \pi \in C_{n+1} : D(\pi) \cap [n-1] = I \} \right| = \left| \{ \sigma \in S_n : D(\sigma) = I \} \right|$. 
Non-bijectione proof using Gessel-Reutenauer

Goal: \[ |\{\pi \in \mathcal{C}_{n+1} : D(\pi) \cap [n-1] = I\}| = |\{\sigma \in S_n : D(\sigma) = I\}|. \]

Let \( I = \{i_1, i_2, \ldots, i_k\} <, I' = I \cup \{n\}. \) By the previous corollary,
\[
|\{\pi \in \mathcal{C}_{n+1} \text{ with } D(\pi) \subseteq I'\}| = |\{\text{necklaces with evaluation } x_1^{i_1} x_2^{i_2-i_1} \ldots x_k^{i_k-i_{k-1}} x_{k+1}^{n-i_k} x_{k+2}\}|.
\]
Non-bijective proof using Gessel-Reutenauer

Goal: \(|\{\pi \in C_{n+1} : D(\pi) \cap [n-1] = I\}| = |\{\sigma \in S_n : D(\sigma) = I\}|\).

Let \(I = \{i_1, i_2, \ldots, i_k\} < \), \(I' = I \cup \{n\}\). By the previous corollary,

\(|\{\pi \in C_{n+1} \text{ with } D(\pi) \cap [n-1] \subseteq I\}| = \left|\{\text{necklaces with evaluation } x_1^{i_1} x_2^{i_2-i_1} \ldots x_k^{i_k-i_{k-1}} x_{k+1}^{n-i_k} x_{k+2}\}\right|.

Non-bijection proof using Gessel-Reutenauer

Goal: $|\{\pi \in C_{n+1} : D(\pi) \cap [n-1] = I\}| = |\{\sigma \in S_n : D(\sigma) = I\}|$.

Let $I = \{i_1, i_2, \ldots, i_k\} <, I' = I \cup \{n\}$. By the previous corollary,

$|\{\pi \in C_{n+1} \text{ with } D(\pi) \cap [n-1] \subseteq I\}| = |\{\text{necklaces with evaluation } x_1^{i_1}x_2^{i_2-i_1} \ldots x_k^{i_k-i_{k-1}}x_{k+1}^{n-i_k}x_{k+2}\}|$.

Choosing first the bead labeled $x_{k+2}$, the # of such necklaces is

$$\binom{n}{i_1, i_2 - i_1, \ldots, i_k - i_{k-1}, n - i_k},$$
Non-bijection proof using Gessel-Reutenauer

**Goal**: \( |\{ \pi \in C_{n+1} : D(\pi) \cap [n-1] = I \}| = |\{ \sigma \in S_n : D(\sigma) = I \}|. \)

Let \( I = \{i_1, i_2, \ldots, i_k\} <, \) \( I' = I \cup \{n\} \). By the previous corollary,
\[
|\{ \pi \in C_{n+1} \text{ with } D(\pi) \cap [n-1] \subseteq I \}| = |\{ \text{necklaces with evaluation } x_1^{i_1} x_2^{i_2-i_1} \ldots x_k^{i_k-i_{k-1}} x_{n-i_k} x_{k+1} x_{k+2} \}|.
\]
Choosing first the bead labeled \( x_{k+2} \), the \# of such necklaces is
\[
\binom{n}{i_1, i_2 - i_1, \ldots, i_k - i_{k-1}, n - i_k},
\]
which is precisely \( |\{ \sigma \in S_n : D(\sigma) \subseteq I \}|. \)
Non-bijection proof using Gessel-Reutenauer

Goal: $|\{\pi \in C_{n+1} : D(\pi) \cap [n-1] = I\}| = |\{\sigma \in S_n : D(\sigma) = I\}|$.

Let $I = \{i_1, i_2, \ldots, i_k\} <, I' = I \cup \{n\}$. By the previous corollary,

$|\{\pi \in C_{n+1} \text{ with } D(\pi) \cap [n-1] \subseteq I\}| = |\{\text{necklaces with evaluation } x_1^{i_1} x_2^{i_2-i_1} \ldots x_k^{i_k-i_{k-1}} x_k^{n-i_k} x_{k+1} x_{k+2}\}|$.

Choosing first the bead labeled $x_{k+2}$, the # of such necklaces is

$$\binom{n}{i_1, i_2 - i_1, \ldots, i_k - i_{k-1}, n - i_k},$$

which is precisely $|\{\sigma \in S_n : D(\sigma) \subseteq I\}|$. We have shown that

$|\{\pi \in C_{n+1} : D(\pi) \cap [n-1] \subseteq I\}| = |\{\sigma \in S_n : D(\sigma) \subseteq I\}|$.

for all $I \subseteq [n-1]$. 
Non-bijection proof using Gessel-Reutenauer

Goal: \( \{\pi \in C_{n+1} : D(\pi) \cap [n-1] = I\} = \{\sigma \in S_n : D(\sigma) = I\} \).

Let \( I = \{i_1, i_2, \ldots, i_k\} \), \( I' = I \cup \{n\} \). By the previous corollary,
\[
|\{\pi \in C_{n+1} \text{ with } D(\pi) \cap [n-1] \subseteq I\}| = |\{\text{necklaces with evaluation } x_1^{i_1} x_2^{i_2-i_1} \ldots x_k^{i_k-i_{k-1}} x_k^{n-i_k} x_{k+1} x_{k+2}\}|.
\]
Choosing first the bead labeled \( x_{k+2} \), the # of such necklaces is
\[
\binom{n}{i_1, i_2 - i_1, \ldots, i_k - i_{k-1}, n - i_k},
\]
which is precisely \( |\{\sigma \in S_n : D(\sigma) \subseteq I\}| \). We have shown that
\[
|\{\pi \in C_{n+1} : D(\pi) \cap [n-1] \subseteq I\}| = |\{\sigma \in S_n : D(\sigma) \subseteq I\}|.
\]
for all \( I \subseteq [n - 1] \). The statement follows by inclusion-exclusion.
An equivalent statement

Let $\mathcal{T}_n$ be the set of $n$-cycles in one-line notation in which one entry has been replaced with 0.

$$\mathcal{T}_3 = \{031, 201, 230, 012, 302, 310\}.$$
An equivalent statement

Let $T_n$ be the set of $n$-cycles in one-line notation in which one entry has been replaced with 0.

$$T_3 = \{031, 201, 230, 012, 302, 310\}.$$

Clearly, $|T_n| = n!$. Descents are defined in the usual way.
An equivalent statement

Let $T_n$ be the set of $n$-cycles in one-line notation in which one entry has been replaced with 0.

$$T_3 = \{031, 201, 230, 012, 302, 310\}.$$  

Clearly, $|T_n| = n!$. Descents are defined in the usual way.

**Corollary**

*For every $n$ there is a bijection between $T_n$ and $S_n$ preserving the descent set.*

Example:

<table>
<thead>
<tr>
<th>$S_3$</th>
<th>123</th>
<th>13·2</th>
<th>2·13</th>
<th>23·1</th>
<th>3·12</th>
<th>3·2·1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_3$</td>
<td>012</td>
<td>03·1</td>
<td>3·02</td>
<td>23·0</td>
<td>3·02</td>
<td>3·1·0</td>
</tr>
</tbody>
</table>
THANK YOU