Derangements and Cubes

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Joint work with Liz McMahon
Problem How many ways can you roll a die so that *none* of its faces are in the same position?

Before            After
Problem: How many ways can you roll a die so that *none* of its faces are in the same position?

Answer: 14

Direct Isometries corresponding to face derangements
Derangements

Hatcheck Problem How many ways can we return $n$ hats to $n$ people so that no one receives her own hat?

A derangement of a set $S$ is a permutation with no fixed points.

**Theorem**

The number of derangements $d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$. Thus, $d_n/n! \to e^{-1} \approx 0.367879 \ldots$

**Theorem**

Recursion: $d_n = (n - 1)(d_{n-1} + d_{n-2})$
Geometry of derangements

**Geometric Fact**

Derangements of $[n] \leftrightarrow$ isometries of the regular $(n - 1)$-simplex in which every one of the $n$ facets is moved.

In $\mathbb{R}^3$, regular tetrahedron has $4!$ isometries – **Rotations**

- **Identity**
- **Face rotations (8)**
- **Edge rotations (3)**
**Geometry of derangements**

**Geometric Fact**  
Derangements of \([n]\) ↔ isometries of the regular \((n - 1)\)-simplex in which every one of the \(n\) facets is moved.

**Reflections and rotary reflections**

**Derangements** 3 edge rotations and 6 rotary reflections: \(d_4 = 9\)
Couples Coatcheck Problem $n$ couples each check their two coats at the beginning of a party; the attendant puts a couple’s 2 coats on a single hanger.

- Attendant randomly selects a hanger;
- Attendant randomly hands a coat from that hanger to each person in the couple.

How many ways can the coats be returned so that no one gets their own coat back?
Cubes and coats

Definition

c-derangements: Let \( \hat{d}_n \) be the number of ways to return the coats so that no one receives their own coat.

Facts:

- There are \( 2^n n! \) ways to return the \( 2n \) coats.
- There are \( 2^n n! \) isometries of an \( n \)-cube.
- The number of coat derangements \( \hat{d}_n \) is the same as the number of facet derangements of the \( n \)-cube.
Motivation
Derangements and geometry

Hypercube derangements and the coatcheck problem

Squares

Deranging the edges of a square.

\( \hat{d}_2 = 5 \)

The 5 edge derangements of a square.
Isometries of the cube

Fact: There are $2^3 \cdot 3! = 48$ isometries of a cube.

- **Direct**
  - The identity;
  - 8 vertex rotations of $120^\circ$ and $240^\circ$;
  - 6 $180^\circ$ edge rotations;
  - 9 rotations through the centers of opposite faces.

- **Indirect**
  - 9 reflections
  - 15 rotary reflections
Direct face derangements

Direct isometries

- The identity;
- 8 vertex rotations of $120^\circ$ and $240^\circ$;
- 6 $180^\circ$ edge rotations;
- 9 rotations through the centers of opposite faces.

Direct Isometries corresponding to face derangements

8 vertex rotations 6 edge rotations
Indirect face derangements

Central inversion \((z \leftrightarrow -z)\)

Reducible rotary reflection \((6)\)

Irreducible rotary reflection \((8)\)

\[
\hat{d}_3 = 14 + 15 = 29
\]
Formulas

**Theorem**

Let \( \hat{d}_n \) be the number of facet derangements of the \( n \)-cube.

\[
\hat{d}_n = 2^n n! \sum_{k=0}^{n} \frac{(-1)^k}{2^k k!}
\]

Compare: \( d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \)

\[\hat{d}_n = \sum_{k=0}^{n} \binom{n}{k} 2^k d_k, \text{ where } d_n = \text{(ordinary) derangements}.\]

Recursion: \( \hat{d}_n = (2n - 1)\hat{d}_{n-1} + (2n - 2)\hat{d}_{n-2} \)

Compare: \( d_n = (n - 1)(d_{n-1} + d_{n-2}) \)
Probabilistic interpretation

_in the coatcheck problem, the probability that no one receives their own coat approaches $e^{-1/2} \approx 0.6065 \ldots$ as $n \to \infty.$

[Compare: $d_n \to e^{-1} \approx 0.3679 \ldots$]

Derangement numbers

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<tbody>
<tr>
<td>$d_n$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>9</td>
<td>44</td>
<td>265</td>
</tr>
<tr>
<td>$\hat{d}_n$</td>
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<td>1</td>
<td>5</td>
<td>29</td>
<td>233</td>
<td>2329</td>
<td>27,949</td>
</tr>
</tbody>
</table>

Rates of convergence

$$\frac{d_6}{6!} - \frac{1}{e} = 1.76 \times 10^{-4}$$

$$\frac{\hat{d}_6}{2^66!} - \frac{1}{\sqrt{e}} = 1.46 \times 10^{-6}$$
Motivation
Derangements and geometry
Hypercube derangements and the coatcheck problem

More data

Ordinary derangements

Direct isometries $\leftrightarrow$ even permutations
Indirect isometries $\leftrightarrow$ odd permutations

Number of even and odd derangements for $n \leq 7$.

<table>
<thead>
<tr>
<th>$n$</th>
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<th>4</th>
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<th>7</th>
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<tr>
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<td>265</td>
<td>1854</td>
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<tr>
<td>$e_n$</td>
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<td>0</td>
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<td>3</td>
<td>24</td>
<td>130</td>
<td>930</td>
</tr>
<tr>
<td>$o_n$</td>
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<td>1</td>
<td>0</td>
<td>6</td>
<td>20</td>
<td>135</td>
<td>924</td>
</tr>
<tr>
<td>$e_n - o_n$</td>
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<td>-1</td>
<td>2</td>
<td>-3</td>
<td>4</td>
<td>-5</td>
<td>6</td>
</tr>
</tbody>
</table>
Hypercube facet derangements

Direct isometries ↔ ‘even’ permutations
Indirect isometries ↔ ‘odd’ permutations

Number of even and odd hypercube derangements for $n \leq 7$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\hat{d}_n$</th>
<th>$\hat{e}_n$</th>
<th>$\hat{o}_n$</th>
<th>$\hat{e}_n - \hat{o}_n$</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>-1</td>
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<tr>
<td>2</td>
<td>5</td>
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<td>4</td>
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</tr>
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<td>7</td>
<td>391,285</td>
<td>195,642</td>
<td>195,643</td>
<td>-1</td>
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</table>
Direct and indirect facet derangements

**Theorem**

Let $\hat{e}_n$ and $\hat{o}_n$ be the number of direct and indirect facet derangements of a cube, resp. Then

$$\hat{e}_n - \hat{o}_n = (-1)^n.$$ 

**Proof idea**

- Each facet derangement $\leftrightarrow$ signed permutation matrix.

$$A = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} \leftrightarrow (11^*)(22^*)(345^*)(3\cdot4\cdot5)$$
\[ \hat{e}_n - \hat{o}_n = (-1)^n. \]

- Easy fact: \( \det(A) = \pm 1. \)

- An isometry is direct iff \( \det(A) = 1. \)

- Find the first row \( k \) with \( a_{k,k} = 0. \)

\[
A = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 & 0 & 0 \\
\end{pmatrix}
\]
\[ \hat{e}_n - \hat{o}_n = (-1)^n. \]

- Change the sign of the only non-zero entry in row \( k \) to produce a new matrix \( A' \):

\[
\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[ A \leftrightarrow (11^*)(22^*)(345^*)(3^*4^*5) \]

\[ A' \leftrightarrow (11^*)(22^*)(34^*53^*45^*) \]
In this example, $A$ is direct and $A'$ is indirect. In general, this involution (almost) gives a 1-1 correspondence between direct and indirect facet-derangements. 

- Central inversion ↔ the matrix $-I$.
- $n$ even ↔ central inversion is direct.
- $n$ odd ↔ central inversion is indirect.
Future projects - 4 dimensions

- Find the number of vertex, edge, 2-dimensional and 3-dimensional face derangement numbers for the 24-cell and the 120-cell.
- For each class of derangements, count the direct and indirect isometries.