Vertex decomposable graphs
and obstructions to shellability

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A simplicial complex $\Delta$ is *shellable* if its facets “fit nicely together”.

A simplicial complex $\Delta$ is *Cohen-Macaulay* if $H^i(\Delta) = 0$ for $i < \dim \Delta$, and if (recursively) every proper link is Cohen-Macaulay.

A simplicial complex $\Delta$ is *sequentially Cohen-Macaulay* if the pure $i$-skeleton (generated by all faces of dimension $i$) is Cohen-Macaulay for every $i$.

Every link of a shellable complex is shellable, and a shellable complex “is” a bouquet of high dimensional spheres, hence $\text{Shellable} \Rightarrow \text{sequentially Cohen-Macaulay}$.
Shellings

A simplicial complex $\Delta$ is *shellable* if its facets “fit nicely together”. Specifically, if there is an ordering $\sigma_1, \ldots, \sigma_m$ of the facets of $\Delta$ such that the intersection of $\sigma_i$ with the union of preceding facets has dimension $(\dim \sigma_i - 1)$. 

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A *shedding vertex* $v$ of a simplicial complex $\Delta$ is such that no face of $\text{link}_\Delta v$ is a facet of $\Delta \setminus v$.

**Lemma:** (Wachs) If $v \in \Delta$ is a shedding vertex, and $\Delta \setminus v$ and $\text{link}_\Delta v$ are shellable, then $\Delta$ is shellable.

**Shelling:** Shelling order of $\Delta \setminus v$ followed by shelling of $v^* \text{link}_\Delta v$.

(So shedding vertex “sorts” facets with $v$ after facets wo/ $v$.)

A complex $\Delta$ is *vertex decomposable* if it is a simplex or (recursively) has a shedding vertex $v$ such that $\Delta \setminus v$ and $\text{link}_\Delta v$ are vertex decomposable.

Vertex decomposable $\implies$ Shellable $\implies$ seq. Cohen-Macaulay
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Shedding vertices and vertex decomposability

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Basic notions

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The *independence complex* of a graph $G = (V, E)$ is the simplicial complex with:

- Vertex set $V$
- Face set $\{\text{independent sets of } G\}$

A complex is *flag* if it is the independence complex of some graph.

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**Vertex decomposable:**
\( \Delta \) a simplex or
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Chordal graphs are vertex decomposable

A graph is *chordal* if it contains no induced cycles of length $> 3$.

Equivalently, every cycle of length $\geq 4$ has a “chord”.

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**Theorem:** (Francisco and Van Tuyl) If $G$ is a chordal graph, then the independence complex of $G$ is sequentially Cohen-Macaulay.

Theorem: (me, Dochtermann-Engström) If $G$ is a chordal graph, then the independence complex of $G$ is vertex decomposable.

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Theorem: (me) If for every independent $A$ in a graph $G$, the subgraph $G \setminus N[A]$ has a “simplicial vertex”, then the independence complex of $G$ is vertex decomposable.
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Chordal graphs are vertex decomposable – sketch

**Shedding vertex** \( v \): independent sets of \( G \setminus N[v] \) are not maximal independent sets of \( G \setminus v \).

**Main fact:** If \( G \) is chordal, then \( G \) has vertex \( w \) with \( N[w] \) a complete subgraph.

Such a \( w \) is called a *simplicial vertex*.

**Lemma:** If \( N[w] \subseteq N[v] \), then \( v \) is a shedding vertex.

**Proof:** Augment any independent set in \( G \setminus N[v] \) by \( w \), giving a larger independent set in \( G \setminus v \).

**Corollary:** Any neighbor of a simplicial vertex is a shedding vertex.

Hence a chordal graph is vertex decomposable.

To show that every link has simplicial vertex \( \Rightarrow \) vertex dec., notice that repeated deletion of neighbors of \( w \) leaves \( w \cup G \setminus N[w] \).
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Simplicial / shedding example

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**Sketch:** A non-trivial theorem of Chvátal, Rusu, and Sritharan says that a graph with no cycles $\geq 6$ which is not the disjoint union of complete graphs has a “3-simplicial path”.

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Shedding vertex $v$: independent sets of $G \setminus N[v]$ are not maximal independent sets of $G \setminus v$. 

![Diagram showing a 3-simplicial path and its effect on vertex shedding](image-url)
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The middle vertex \( v \) of a 3-simplicial path is a shedding vertex:
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The middle vertex $v$ of a 3-simplicial path is a shedding vertex: An independent set in $G \setminus N[v]$ can be augmented by either $w_1$ or $w_2$, since it can’t neighbor both of them.
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Corollary: (me) The obstructions to shellability (minimal non-shellable complexes) in flag complexes are exactly the independence complexes of $C_n$, $n \neq 3, 5$. 
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Part 2: Clutters
Flag complexes can be described in terms of their facets (maximal faces), or in terms of their minimal non-faces.

A general simplicial complex can also be described in terms of minimal non-faces. The non-faces can be any set system $C$, with the restriction that $X, Y \in C \Rightarrow X \not\subseteq Y$.

This is a kind of set system, called a clutter or Sperner system.

Can we relate the clutter-theoretic properties of $C$ to shellability of its independence complex?
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Graphs → Clutters

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Can we relate the clutter-theoretic properties of $\mathcal{C}$ to shellability of its independence complex?
Chordal clutters

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**Theorem:** (me) The independence complex of a chordal clutter is shellable.
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**Application:** there are 21 obstructions to shellability on 6 vertices that have every link shellable. (by GAP computation)
Reference:


Thank you!

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