Conjectures concerning the difference of two skew Schur functions

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www.facstaff.bucknell.edu/pm040/
The setting

$s_A$: the skew Schur function for the skew shape $A$

**Overarching Question.** For skew shapes $A$ and $B$, when is $s_A - s_B$ Schur-positive?

Want simple conditions in terms of the shapes of $A$ and $B$. 

Conjectures on differences of skew Schurs

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The setting

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**Overarching Question.** For skew shapes \( A \) and \( B \), when is

\[ s_A - s_B \]

Schur-positive?

Want simple conditions in terms of the shapes of \( A \) and \( B \).

**Special Case.** For partitions \( \alpha, \beta, \gamma, \delta \), when is

\[ s_\alpha s_\beta - s_\gamma s_\delta \]

Schur-positive?

\[ \begin{array}{c|c|c|c|c|c|c|c|c} \hline \hline & & & & & & & & \\ \hline & & & & & & \cdot \hspace{1cm} \cdot & & \\ \hline \hline \end{array} \hspace{.5cm} = \hspace{.5cm} \begin{array}{c|c|c|c|c|c|c|c|c} \hline \hline & & \cdot & & \cdot & & \cdot & & \cdot \\ \hline \cdot & & & & & & & & \cdot \\ \hline \hline \end{array} \]

[Azenhas, Ballantine, F. Bergeron, Biagioli, Conflitti, Fomin, Fulton, King, A. N. Kirillov, Lam, Lascoux, Leclerc, C.-K. Li, Mamede, M., Okounkov, Orellana, Poon, Postnikov, Pylyavskyy, Rosas, Thibon, Welsh, van Willigenburg, ...]
The problems and conjectures

1. Equality of skew Schur functions
   \textit{Joint with Stephanie van Willigenburg}

2. Connected skew Schur functions maximal in Schur-positivity order
   \textit{Joint with Pavlo Pylyavskyy and Stephanie van Willigenburg}

3. $F$-support containment and the row-overlap conditions of Reiner, Shaw and van Willigenburg

4. A Saturation Theorem for skew Schur functions
   \textit{Joint with Alejandro Morales}
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1. Equality of skew Schur functions

Problem 1. When is $s_A = s_B$? Denoted $A \sim B$.

Determine necessary and sufficient conditions on shapes of $A$ and $B$.

Lou Billera, Hugh Thomas, Steph van Willigenburg (2004): complete answer for ribbons
John Stembridge (2004): skewed staircases
Vic Reiner, Kristin Shaw, Steph van Willigenburg (2006): 3 operations for generating skew shapes with equal skew Schur functions; necessary conditions
M., Steph van Willigenburg (2006): unification, generalization, conjecture for necessary and sufficient conditions
Christian Gutschwager (2008): multiplicity-free skew shapes

Conjectures on differences of skew Schurs

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\[ \begin{array}{c c c c c}
\circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ \\
\end{array} \sim \\
\begin{array}{c c c c c}
\circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ \\
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1. Equality of skew Schur functions

(With apologies)

Conjecture 1 [M., van Willigenburg (2006); inspired by main result of BTvW (2006)].

Two skew shapes $E$ and $E'$ satisfy $E \sim E'$ if and only if, for some $r$,

\[
E = (((\cdots (E_1 \circ W_2 E_2) \circ W_3 E_3) \cdots) \circ W_r E_r
\]

\[
E' = (((\cdots (E'_1 \circ W'_2 E'_2) \circ W'_3 E'_3) \cdots) \circ W_r E'_r ,
\]

where

- $E_i = W_i O_i W_i$ satisfies four hypotheses for all $i$,
- $E'_i$ and $W'_i$ denote either $E_i$ and $W_i$, or $E_i^*$ and $W_i^*$. 

1. Equality of skew Schur functions

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Conjecture 1 [M., van Willigenburg (2006); inspired by main result of BTvW (2006)].
Two skew shapes $E$ and $E'$ satisfy $E \sim E'$ if and only if, for some $r$,

$$E = (((\cdots (E_1 \circ W_2 E_2) \circ W_3 E_3) \cdots) \circ W_r) E_r$$
$$E' = (((\cdots (E'_1 \circ W'_2 E'_2) \circ W'_3 E'_3) \cdots) \circ W_r) E'_r,$$

where

- $E_i = W_i O_i W_i$ satisfies four hypotheses for all $i$,
- $E'_i$ and $W'_i$ denote either $E_i$ and $W_i$, or $E_i^*$ and $W_i^*$.


- With one more hypothesis, the “if” direction
- $n \leq 20$

Evidence [Gutschwager, 2006]. Multiplicity-free skew shapes
2. Maximal connected skew shapes

Definition. Let $A, B$ be skew shapes. We say that $A \geq s B$ if $s A - s B$ is Schur-positive. If $B \leq s A$ then $|A| = |B|$.

Example. $P_4$:

Problem 2. What are the maximal elements of $P_n$ among the connected skew shapes?
2. Maximal connected skew shapes

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$$A \geq_s B \text{ if } s_A - s_B \text{ is Schur-positive.}$$

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**Example.** $P_4$:

![Diagram of skew shapes]

Conjectures on differences of skew Schurs

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Example. $P_4$:

Problem 2. What are the maximal elements of $P_n$ among the connected skew shapes?
2. Maximal connected skew shapes

**Conjecture 2 [M., Pylyavskyy (2007)].** For each $r = 1, \ldots, n$, there is a unique maximal connected element with $r$ rows, namely the ribbon marked out by the diagonal of an $r$-by-$(n - r + 1)$ box.

**Examples.**

![Diagram of maximal connected skew shapes with ribbons marked out by diagonals.]

**Evidence [M., van Willigenburg (2011)].**

- Maximal element must be an equitable ribbon: row (resp. column) lengths differ by at most 1.
- $\text{Supp}^A B \{ \lambda \vdash n \mid s_\lambda \text{appears in the Schur expansion of } s_A \},$ the Schur-support of $A$.
- $\text{Supp}^s = \{ 3, 21, 111 \}$.
- True in Support Poset: $A \geq \text{Supp}^s B$ if $\text{Supp}^A \supseteq \text{Supp}^B$. 
2. Maximal connected skew shapes

Conjecture 2 [M., Pylyavskyy (2007)]. For each \( r = 1, \ldots, n \), there is a unique maximal connected element with \( r \) rows, namely the ribbon marked out by the diagonal of an \( r \)-by-\((n - r + 1) \) box.

Examples.

Evidence [M., van Willigenburg (2011)].

- \( n \leq 34 \)
- Maximal element must be an equitable ribbon: row (resp. column) lengths differ by at most 1.
- \( \text{Supp}_s(A) := \{ \lambda \vdash n \mid s_\lambda \text{ appears in the Schur expansion of } s_A \} \), the Schur-support of \( A \).
  e.g. \( s_{d^3} = s_3 + 2s_{21} + s_{111} \). \( \text{Supp}_s(d^3) = \{3, 21, 111\} \).

True in Support Poset: \( A \supseteq_{\text{Supp}_s} B \) if \( \text{Supp}_s(A) \supseteq \text{Supp}_s(B) \).
3. The row-overlap conditions

General idea: the overlaps among rows must match up for $A = B$.

Definition [Reiner, Shaw, van Willigenburg]. For a skew shape $A$, let $\text{overlap}_k(i)$ be the number of columns occupied in common by rows $i, i+1, \ldots, i+k-1$.

Then rows $\text{rows}_k(A)$ is the weakly decreasing rearrangement of $(\text{overlap}_k(1), \text{overlap}_k(2), \ldots)$.

Example.

$A = \begin{array}{cccc}
& & & \\
& & 1 & \\
& 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{array}$

$\text{overlap}_1(i) = \text{length of the } i\text{th row}. \text{ Thus } \text{rows}_1(A) = 44211.$

$\text{overlap}_2(1) = 2, \text{overlap}_2(2) = 3, \text{overlap}_2(3) = 1, \text{overlap}_2(4) = 1, \text{ so } \text{rows}_2(A) = 3211.$

$\text{rows}_3(A) = 11.$

$\text{rows}_k(A) = \emptyset$ for $k > 3.$
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**Definition [Reiner, Shaw, van Willigenburg].** For a skew shape $A$, let $\text{overlap}_k(i)$ be the number of columns occupied in common by rows $i$, $i+1$, ..., $i+k-1$.

Then rows $k(A)$ is the weakly decreasing rearrangement of $(\text{overlap}_1(1), \text{overlap}_1(2), ...)$.

**Example.**

$A = ▶\text{overlap}_1(1) = \text{length of the } i\text{th row. Thus rows } 1(A) = 44211.$

$\text{overlap}_2(1) = 2, \text{overlap}_2(2) = 3, \text{overlap}_2(3) = 1, \text{overlap}_2(4) = 1,$ so rows $2(A) = 3211.$

$\text{rows } 3(A) = 11.$

$\text{rows } k(A) = \emptyset$ for $k > 3.$
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**Definition [Reiner, Shaw, van Willigenburg].** For a skew shape $A$, let $\text{overlap}_k(i)$ be the number of columns occupied in common by rows $i, i + 1, \ldots, i + k - 1$. Then $\text{rows}_k(A)$ is the weakly decreasing rearrangement of $(\text{overlap}_k(1), \text{overlap}_k(2), \ldots)$.

**Example.**

\[
A = \begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{array}
\]
3. The row-overlap conditions

General idea: the overlaps among rows must match up for $s_A = s_B$.

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Then $\text{rows}_k(A)$ is the weakly decreasing rearrangement of $(\text{overlap}_k(1), \text{overlap}_k(2), \ldots)$.

Example.

\[
A = \begin{array}{cccc}
\square & \square & \square & \square \\
\square & \square & \square & \square \\
\square & \square & \square & \square \\
\square & \square & \square & \square \\
\square & \square & \square & \square \\
\end{array}
\]

- $\text{overlap}_1(i) =$ length of the $i$th row. Thus $\text{rows}_1(A) = 44211$. 
3. The row-overlap conditions

General idea: the overlaps among rows must match up for \( s_A = s_B \).

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Then \( \text{rows}_k(A) \) is the weakly decreasing rearrangement of \( (\text{overlap}_k(1), \text{overlap}_k(2), \ldots) \).

Example.

\[
A = \begin{array}{cccc}
\text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } \\
\end{array}
\]

\( \text{overlap}_1(i) = \) length of the \( i \)th row. Thus \( \text{rows}_1(A) = 44211 \).

\( \text{overlap}_2(1) = 2, \text{overlap}_2(2) = 3, \text{overlap}_2(3) = 1, \text{overlap}_2(4) = 1, \) so \( \text{rows}_2(A) = 3211 \).
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Then \( \text{rows}_k(A) \) is the weakly decreasing rearrangement of \((\text{overlap}_k(1), \text{overlap}_k(2), \ldots)\).

Example.

\[
A = \\
\begin{array}{cccc}
\text{row} & \text{row} & \text{row} & \text{row} \\
\text{row} & \text{row} & \text{row} & \text{row} \\
\text{row} & \text{row} & \text{row} & \text{row} \\
\text{row} & \text{row} & \text{row} & \text{row} \\
\end{array}
\]

- \( \text{overlap}_1(i) = \) length of the \( i \)th row. Thus \( \text{rows}_1(A) = 44211 \).
- \( \text{overlap}_2(1) = 2, \text{overlap}_2(2) = 3, \text{overlap}_2(3) = 1, \text{overlap}_2(4) = 1, \) so \( \text{rows}_2(A) = 3211 \).
- \( \text{rows}_3(A) = 11 \).
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Example.

\[
A = \begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

- $\text{overlap}_1(i) = \text{length of the } i\text{th row. Thus } \text{rows}_1(A) = 44211.$
- $\text{overlap}_2(1) = 2, \text{overlap}_2(2) = 3, \text{overlap}_2(3) = 1,$
  $\text{overlap}_2(4) = 1,$ so $\text{rows}_2(A) = 3211.$
- $\text{rows}_3(A) = 11.$
- $\text{rows}_k(A) = \emptyset$ for $k > 3.$
3. The row-overlap conditions

Necessary conditions for equality

**Theorem [RSvW, (2006)].** Let $A$ and $B$ be skew shapes. If $s_A = s_B$, then

$$\text{rows}_k(A) = \text{rows}_k(B) \text{ for all } k.$$ 

**Question.** What are necessary conditions on $A$ and $B$ for $s_A - s_B$ to be Schur-positive?

**Theorem [M., (2008)].** Let $A$ and $B$ be skew shapes. If $s_A - s_B$ is Schur-positive, then

$$\text{rows}_k(A) \leq_{\text{dom}} \text{rows}_k(B) \text{ for all } k.$$
3. The row-overlap conditions

**Necessary conditions for equality**

**Theorem [RSvW, (2006)].** Let $A$ and $B$ be skew shapes. If $s_A = s_B$, then

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$$\text{rows}_k(A) \leq_{\text{dom}} \text{rows}_k(B) \text{ for all } k.$$ 

In fact, it suffices to assume that $\text{Supp}_s(A) \supseteq \text{Supp}_s(B)$. 

Conjectures on differences of skew Schurs  

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### 3. The row-overlap conditions

**Theorem [M., (2008)].**

\[ s_A - s_B \text{ is Schur-pos.} \implies \text{Supp}_s(A) \supseteq \text{Supp}_s(B) \]

Equivalent choices:

\[ \text{rows}_k(A) \leq_{\text{dom}} \text{rows}_k(B) \quad \forall k \]

\[ \text{cols}_\ell(A) \leq_{\text{dom}} \text{cols}_\ell(B) \quad \forall \ell \]

\[ \text{rects}_{k,\ell}(A) \leq \text{rects}_{k,\ell}(B) \quad \forall k, \ell \]

Converse is already false at \( n = 4 \).

Problem 3. What weaker algebraic conditions best fill the gap?

Conjecture 4 [M., (2013)]. The rightmost implication is if and only if.

Evidence [M., (2013)]. Conjecture is true for:

- \( n \leq 13 \) (compare with failure at \( n = 4 \) for other converse implications)
- \( F \)-multiplicity-free skew shapes (as classified by Christine Bessenrodt and Steph van Willigenburg, (2013));
- ribbons whose rows all have length at least 2.
3. The row-overlap conditions

Theorem [M., (2008)].

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\[ \text{rows}_k(A) \leq_{\text{dom}} \text{rows}_k(B) \forall k \]
\[ \text{cols}_\ell(A) \leq_{\text{dom}} \text{cols}_\ell(B) \forall \ell \]
\[ \text{rects}_{k,\ell}(A) \leq \text{rects}_{k,\ell}(B) \forall k, \ell \]
3. The row-overlap conditions and $F$-support containment

**Theorem [M., (2013)].**

\[
\begin{align*}
\text{s}_A - \text{s}_B \text{ is Schur-pos.} \quad &\Rightarrow \quad \text{Supp}_s(A) \supseteq \text{Supp}_s(B) \\
\downarrow &\Rightarrow \\
\text{s}_A - \text{s}_B \text{ is } F\text{-positive} \quad &\Rightarrow \quad \text{Supp}_F(A) \supseteq \text{Supp}_F(B) \\
\downarrow &\Rightarrow \\
\text{rows}_k(A) \leq_{\text{dom}} \text{rows}_k(B) \quad &\forall k \\
\text{cols}_\ell(A) \leq_{\text{dom}} \text{cols}_\ell(B) \quad &\forall \ell \\
\text{rects}_{k,\ell}(A) \leq \text{rects}_{k,\ell}(B) \quad &\forall k, \ell
\end{align*}
\]

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- rows$_k(A) \leq_{\text{dom}}$ rows$_k(B) \quad \forall k$
- cols$_\ell(A) \leq_{\text{dom}}$ cols$_\ell(B) \quad \forall \ell$
- rects$_{k,\ell}(A) \leq$ rects$_{k,\ell}(B) \quad \forall k, \ell$

**Problem 3.** What weaker algebraic conditions best fill the gap?
3. The row-overlap conditions and \( F \)-support containment

**Theorem [M., (2013)].**

\[
\begin{align*}
s_A - s_B \text{ is Schur-pos.} & \quad \Rightarrow \quad \text{Supp}_S(A) \supseteq \text{Supp}_S(B) \\
\downarrow & \\
s_A - s_B \text{ is } F\text{-positive} & \quad \Rightarrow \quad \text{Supp}_F(A) \supseteq \text{Supp}_F(B)
\end{align*}
\]

Problem 3. What weaker algebraic conditions best fill the gap?

**Conjecture 4 [M., (2013)].** The rightmost implication is if and only if.

\[
\begin{align*}
\text{Converse is already false at } n = 4. \\
\text{Evidence [M., (2013)]. Conjecture is true for:} \\
\text{\hspace{1cm}▶ } n \leq 13 \text{ (compare with failure at } n = 4 \text{ for other converse implications)} \\
\text{\hspace{1cm}▶ } F\text{-multiplicity-free skew shapes (as classified by Christine Bessenrodt and Steph van Willigenburg, (2013))} \\
\text{\hspace{1cm}▶ } \text{ribbons whose rows all have length at least 2.}
\end{align*}
\]
3. The row-overlap conditions and $F$-support containment

**Theorem [M., (2013)].**

\[
\begin{align*}
\text{s}_A - \text{s}_B \text{ is Schur-pos.} & \quad \Rightarrow \quad \text{Supp}_s(A) \supseteq \text{Supp}_s(B) \\
\downarrow & \\
\text{s}_A - \text{s}_B \text{ is } F\text{-positive} & \quad \Rightarrow \quad \text{Supp}_F(A) \supseteq \text{Supp}_F(B)
\end{align*}
\]

Equivalent choices:

\[
\begin{align*}
\text{rows}_k(A) \leq_{\text{dom}} \text{rows}_k(B) & \quad \forall k \\
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**Problem 3.** What weaker algebraic conditions best fill the gap?

**Conjecture 4 [M., (2013)].** The rightmost implication is if and only if.

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- $F$-multiplicity-free skew shapes (as classified by Christine Bessenrodt and Steph van Willigenburg, (2013));
- ribbons whose rows all have length at least 2.
3. The row-overlap conditions and $F$-support containment

Example. $n = 6$

$F$-support containment

Dual of row overlap dominance
3. The row-overlap conditions and $F$-support containment

**Example.** $n = 12$ case has 12,042 edges.
4. A Saturation Theorem for skew Schur functions

A = λ/µ and k is a positive integer, define kA = kλ/kµ.

Theorem [Knutson, Tao, (1999)]. For a skew shape $A$ and partition $\nu$, $\nu \in \text{Supp } s(A) \iff n\nu \in \text{Supp } s(nA)$.

Equivalently, $\text{Supp } s(\nu) \subseteq \text{Supp } s(A) \iff \text{Supp } s(n\nu) \subseteq \text{Supp } s(nA)$.

Problem 4. [Speyer (2009)]. Can this be generalized by replacing $\nu$ by a skew shape?

Answer. No. False even in the easier direction $\Rightarrow$.

Conjecture 4 [M., Morales (2014)]. A quasisymmetric skew Saturation Theorem: $\text{Supp } F(B) \subseteq \text{Supp } F(A) \iff \text{Supp } F(nB) \subseteq \text{Supp } F(nA)$.

Evidence. Follows from Conjecture 3.
4. A Saturation Theorem for skew Schur functions

$A = \lambda/\mu$ and $k$ is a positive integer, define $kA = k\lambda/k\mu$.

Theorem [Knutson, Tao, (1999)]. For a skew shape $A$ and partition $\nu$,

$\nu \in \text{Supp}_s(A) \iff n\nu \in \text{Supp}_s(nA)$.

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\( A = \lambda/\mu \) and \( k \) is a positive integer, define \( kA = k\lambda/k\mu \).

Theorem [Knutson, Tao, (1999)]. For a skew shape \( A \) and partition \( \nu \),

\[ \nu \in \text{Supp}_s(A) \iff n\nu \in \text{Supp}_s(nA). \]

Equivalently,

\[ \text{Supp}_s(\nu) \subseteq \text{Supp}_s(A) \iff \text{Supp}_s(n\nu) \subseteq \text{Supp}_s(nA). \]

Problem 4. [Speyer (2009)]. Can this be generalized by replacing \( \nu \) by a skew shape?

Answer. No. False even in the easier direction \( \implies \).

Conjecture 4 [M., Morales (2014)]. A quasisymmetric skew Saturation Theorem:

\[ \text{Supp}_s(B) \subseteq \text{Supp}_s(A) \iff \text{Supp}_s(nB) \subseteq \text{Supp}_s(nA). \]

Evidence. Follows from Conjecture 3.
4. A Saturation Theorem for skew Schur functions

$A = \lambda/\mu$ and $k$ is a positive integer, define $kA = k\lambda/k\mu$.

Theorem [Knutson, Tao, (1999)]. For a skew shape $A$ and partition $\nu$,

$$\nu \in \text{Supp}_s(A) \iff n\nu \in \text{Supp}_s(nA).$$

Equivalently,

$$\text{Supp}_s(\nu) \subseteq \text{Supp}_s(A) \iff \text{Supp}_s(n\nu) \subseteq \text{Supp}_s(nA).$$

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**Evidence.** Follows from Conjecture 3.
1. Equality of skew Schur functions

Conjecture 1 [M., van Willigenburg (2006); inspired by main result of BTvW (2006)].

Two skew shapes $E$ and $E'$ satisfy $E \sim E'$ if and only if, for some $r$,

$$E = (((E_1 \circ W_2 E_2) \circ W_3 E_3) \circ \ldots \circ W_r E_r)$$

$$E' = (((E'_1 \circ W'_2 E'_2) \circ W'_3 E'_3) \circ \ldots \circ W'_r E'_r),$$

where $\circ E_i = W_i O_i W_i$ satisfies four hypotheses for all $i$, $\circ E'_i$ and $W'_i$ denote either $E_i$ and $W_i$, or $E^*_i$ and $W^*_i$. 
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Two skew shapes $E$ and $E'$ satisfy $E \sim E'$ if and only if, for some $r$,

$$
E = ((( \cdots (E_1 \circ_{W_2} E_2) \circ_{W_3} E_3 ) \cdots ) \circ_{W_r} E_r
$$

$$
E' = ((( \cdots (E'_1 \circ_{W'_2} E'_2) \circ_{W'_3} E'_3 ) \cdots ) \circ_{W_r} E'_r
$$

where

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- $E'_i$ and $W'_i$ denote either $E_i$ and $W_i$, or $E_i^*$ and $W_i^*$. 
1. Equality of skew Schur functions

Composition of skew shapes

\[ D \circ E = \] 

\[ \quad \circ \quad = \] 

\[ D \circ E \] 

Theorem [M., van Willigenburg, (2006)]. If \( D \sim D' \), then \( D' \circ E \sim D \circ E \sim D \circ E^* \).
1. Equality of skew Schur functions

Composition of skew shapes

\[ D \circ E = \begin{array}{ccc}
\text{\textcolor{blue}{\begin{array}{ccc}
\text{\textcolor{red}{\begin{array}{ccc}
\text{\textcolor{green}{\begin{array}{ccc}
\text{\textcolor{purple}{\begin{array}{ccc}
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\end{array} \end{array} \]
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Composition of skew shapes

\[ D \circ E = \begin{array}{ccc}
\text{\color{red}{\square}} & \text{\color{green}{\square}} & \text{\color{blue}{\square}} \\
\text{\color{red}{\square}} & \text{\color{red}{\square}}
\end{array} \circ \begin{array}{ccc}
\text{\color{red}{\square}} & \text{\color{red}{\square}} & \text{\color{red}{\square}} & \text{\color{red}{\square}} \\
\text{\color{red}{\square}} & \text{\color{red}{\square}} & \text{\color{red}{\square}} & \text{\color{red}{\square}} & \text{\color{red}{\square}} \\
\text{\color{red}{\square}} & \text{\color{red}{\square}} & \text{\color{red}{\square}} & \text{\color{red}{\square}} & \text{\color{red}{\square}} & \text{\color{red}{\square}} \\
\end{array} = \begin{array}{ccc}
\text{\color{red}{\square}} & \text{\color{red}{\square}} & \text{\color{red}{\square}} & \text{\color{red}{\square}} & \text{\color{red}{\square}} & \text{\color{red}{\square}} \\
\text{\color{red}{\square}} & \text{\color{red}{\square}} & \text{\color{red}{\square}} & \text{\color{red}{\square}} & \text{\color{red}{\square}} & \text{\color{red}{\square}} & \text{\color{red}{\square}} \\
\text{\color{red}{\square}} & \text{\color{red}{\square}} & \text{\color{red}{\square}} & \text{\color{red}{\square}} & \text{\color{red}{\square}} & \text{\color{red}{\square}} & \text{\color{red}{\square}} \\
\end{array} \]
1. Equality of skew Schur functions

Composition of skew shapes

\[ D \circ E = \begin{array}{cccc}
\square & \square & \square & \square \\
\square & \square & & \\
\end{array} \circ \begin{array}{cc}
\square & \square \\
\end{array} = \begin{array}{cccc}
\square & \square & \square & \square \\
\square & \square & & \\
\square & \square & & \\
\square & \square & & \\
\end{array} \]
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Composition of skew shapes

\[ D \circ E = \begin{array}{cccc} & & & X \\ & & & \end{array} \circ \begin{array}{cccc} & & & \\ & & & \end{array} = \begin{array}{cccc} & & & X \\ & & & \end{array} \]

Theorem [M., van Willigenburg, (2006)]. If \( D \sim D' \), then

\[ D' \circ E \sim D \circ E \sim D \circ E^*. \]
1. Equality of skew Schur functions

Amalgamated compositions: $\circ_W$

A skew shape $W$ lies in the top of a skew shape $E$ if $W$ appears as a connected subshape of $E$ that includes the northeasternmost cell of $E$.

Similarly, $W$ lies in the bottom of $E$. Our interest.

$W$ lies in both the top and bottom of $E$. We write $E = W \circ W$.

Hypotheses [inspired by hypotheses of RSvW].

1. $W_1$ and $W_2$ are separated by at least one diagonal.
2. $E \setminus W_1$ and $E \setminus W_2$ are both connected skew shapes.
3. $W$ is maximal given its set of diagonals.
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Hypotheses [inspired by hypotheses of RSvW].
1. $W_{ne}$ and $W_{sw}$ are separated by at least one diagonal.
2. $E \setminus W_{ne}$ and $E \setminus W_{sw}$ are both connected skew shapes.
3. $W$ is maximal given its set of diagonals.
Example.

\[ D \circ_W E = \]

\[ = \]

Conjectures on differences of skew Schurs
1. Equality of skew Schur functions

Example.

\[ D \circ_w E = \]

\[ = \]

\[ = \]
1. Equality of skew Schur functions

Example.

\[ D \circ_w E \triangleq \begin{array}{ccc}
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\text{\ }
\end{array}
\end{array} \]

\[ \begin{array}{ccc}
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\[ = \]

\[ \begin{array}{ccc}
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\[ \begin{array}{ccc}
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\[ \begin{array}{ccc}
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\[ \begin{array}{ccc}
\begin{array}{c}
\text{\ }
\end{array}
\end{array} \]

Conjectures on differences of skew Schurs

Peter McNamara
Example.

\[ D \circ_W E = \]

\[ \circ \]

\[ = \]

\[ \]

Conjectures on differences of skew Schurs

Peter McNamara
Example.

\[ D \circ_W E = \quad \circ \quad = \quad \sim \]
1. Equality of skew Schur functions

Construction of $\overline{W}$ and $\overline{O}$:

Hypothesis 4. $W$ is never adjacent to $O$.

Conjecture 1. Two skew shapes $E$ and $E'$ satisfy $E \sim E'$ if and only if, for some $r$, $E = (((E_1 \circ W_2 E_2) \circ W_3 E_3) \cdots) \circ W_r E_r = (((E'_1 \circ W'_2 E'_2) \circ W'_3 E'_3) \cdots) \circ W'_r E'_r$, where $\circ E_i = W_i O_i W_i$ satisfies Hypotheses 1–4 for all $i$.

Thanks!
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$$

$$
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