Comparing skew Schur functions: a quasisymmetric perspective

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Slides and paper available from
www.facstaff.bucknell.edu/pm040/
The background story: the equality question

Conditions for Schur-positivity

Quasisymmetric insights and the big conjecture

Relationship to other (quasi)symmetric bases
Comparing skew Schur functions quasisymmetrically

Peter McNamara
The skew Schur function \( s_A \) in the variables \( x = (x_1, x_2, \ldots) \) is then defined by

\[
s_A = \sum_{\text{SSYT } T} x_1^{\#1's \text{ in } T} x_2^{\#2's \text{ in } T} \ldots.
\]

Example.

\[
s_{4431/31} = x_4^3 x_5 x_6^2 x_7 x_9 + \cdots.
\]
Question. When is $s_A = s_B$?

1. 

2. 

3. 

4. 

Definition. A ribbon is a connected skew shape containing no $2 \times 2$ rectangle.
The equality question

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The equality question

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Open Problem. Find necessary and sufficient conditions on $A$ and $B$ for $s_A = s_B$. 
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  Complete classification of equality of ribbon Schur functions.
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  Complete classification of equality of ribbon Schur functions.

- McN., Steph van Willigenburg (2006)
- Christian Gutschwager (2008) solved multiplicity-free case

Open Problem. Find necessary and sufficient conditions on $A$ and $B$ for $s_A = s_B$. 
Necessary conditions for equality

General idea: the overlaps among rows must match up.

Definition [Reiner, Shaw, van Willigenburg]. For a skew shape $A$, let $\text{overlap}_k(i)$ be the number of columns occupied in common by rows $i, i+1, \ldots, i+k-1$.

Then rows $\text{rows}_k(A)$ is the weakly decreasing rearrangement of $(\text{overlap}_1(1), \text{overlap}_1(2), \ldots)$. 

Example. $A = \begin{array}{cccc} & & & \\ & & & \\ & & & \\ & & & \\ & & & \end{array}$ $\text{overlap}_1(1) = \text{length of the } i\text{th row.}$ Thus rows $\text{rows}_1(A) = 44211$.

$\text{overlap}_2(1) = 2$, $\text{overlap}_2(2) = 3$, $\text{overlap}_2(3) = 1$, $\text{overlap}_2(4) = 1$, so rows $\text{rows}_2(A) = 3211$.

$\text{rows}_3(A) = 11$. $\text{rows}_k(A) = \emptyset$ for $k > 3$. 
Necessary conditions for equality

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**Example.**

$$A = \begin{array}{c}
\text{1} & \text{2} & \text{3} \\
\text{4} & \text{5} & \text{6} \\
\text{7} & \text{8} & \text{9} \\
\end{array}$$
Necessary conditions for equality

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**Definition [Reiner, Shaw, van Willigenburg].** For a skew shape \( A \), let \( \text{overlap}_k(i) \) be the number of columns occupied in common by rows \( i, i + 1, \ldots, i + k - 1 \).

Then rows\(_k(A)\) is the weakly decreasing rearrangement of \((\text{overlap}_k(1), \text{overlap}_k(2), \ldots)\).

**Example.**

\[
A = \begin{array}{cccc}
\blacksquare & \blacksquare & \blacksquare & \blacksquare \\
& \blacksquare & \blacksquare & \\
& & \blacksquare & \\
& & & \\
\end{array}
\]

- \( \text{overlap}_1(i) = \) length of the \( i \)th row. Thus rows\(_1(A) = 44211 \).
Necessary conditions for equality

**General idea:** the overlaps among rows must match up.

**Definition [Reiner, Shaw, van Willigenburg].** For a skew shape $A$, let $\text{overlap}_k(i)$ be the number of columns occupied in common by rows $i, i+1, \ldots, i+k-1$.

Then $\text{rows}_k(A)$ is the weakly decreasing rearrangement of $(\text{overlap}_k(1), \text{overlap}_k(2), \ldots)$.

**Example.**

\[
A = \begin{array}{cccc}
\square & \square & \square & \square \\
\square & \square & \square & \square \\
\end{array}
\]

- $\text{overlap}_1(i) =$ length of the $i$th row. Thus $\text{rows}_1(A) = 44211$.
- $\text{overlap}_2(1) = 2$, $\text{overlap}_2(2) = 3$, $\text{overlap}_2(3) = 1$, $\text{overlap}_2(4) = 1$, so $\text{rows}_2(A) = 3211$. 

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Comparing skew Schur functions quasisymmetrically

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Example.

\[
A = \begin{array}{cccc}
\text{x} & \text{x} & \text{x} & \text{x} \\
\text{x} & \text{x} & \text{x} & \text{x} \\
\text{x} & \text{x} & \text{x} & \text{x} \\
\text{x} & \text{x} & \text{x} & \text{x} \\
\end{array}
\]

- $\text{overlap}_1(i) = \text{length of the } i\text{th row. Thus } \text{rows}_1(A) = 44211.$
- $\text{overlap}_2(1) = 2, \text{overlap}_2(2) = 3, \text{overlap}_2(3) = 1,$
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Example.

\[ A = \begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
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- $\text{overlap}_1(i) = \text{length of the } i\text{th row. Thus } \text{rows}_1(A) = 44211$.
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- $\text{rows}_3(A) = 11$.
- $\text{rows}_k(A) = \emptyset$ for $k > 3$. 
Theorem [RSvW]. Let $A$ and $B$ be skew shapes. If $s_A = s_B$, then

$$\text{rows}_k(A) = \text{rows}_k(B) \text{ for all } k.$$
Necessary conditions for equality

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Converse is not true:
Our interest: inequalities.

\[ s_{\lambda/\mu} = \sum_{\nu} c_{\mu\nu}^{\lambda} s_\nu. \]

When is \( s_{\lambda/\mu} - s_{\sigma/\tau} \) Schur-positive?
Schur-positivity order

Our interest: inequalities.

\[ s_{\lambda/\mu} = \sum_{\nu} c_{\mu\nu}^{\lambda} s_{\nu}. \]

When is \( s_{\lambda/\mu} - s_{\sigma/\tau} \) Schur-positive?

Definition. Let \( A, B \) be skew shapes. We say that

\[ A \geq_s B \quad \text{if} \quad s_A - s_B \quad \text{is Schur-positive}. \]

Original goal: characterize the Schur-positivity order \( \geq_s \) in terms of skew shapes.
Example of a Schur-positivity poset

If \( B \leq_s A \) then \(|A| = |B|\).
Call the resulting ordered set \( P_n \).
Then \( P_4 \):
More examples

$P_5$: 

$P_6$: 

Comparing skew Schur functions quasisymmetrically  

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Known properties: Sufficient conditions

Sufficient conditions for $A \succeq_s B$:

- Alain Lascoux, Bernard Leclerc, Jean-Yves Thibon (1997)
- Andrei Okounkov (1997)
- Sergey Fomin, William Fulton, Chi-Kwong Li, Yiu-Tung Poon (2003)
- Thomas Lam, Alex Postnikov, Pavlo Pylyavskyy (2005)
- François Bergeron, Riccardo Biagioli, Mercedes Rosas (2006)
- ...
Necessary conditions for Schur-positivity

Notation. Write \( \lambda \preceq \mu \) if \( \lambda \) is less than or equal to \( \mu \) in dominance order, i.e.

\[
\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i \quad \text{for all} \quad i.
\]

Theorem [McN. (2008)]. Let \( A \) and \( B \) be skew shapes. If \( s_A - s_B \) is Schur-positive, then \( \text{rows} \ k(A) \preceq \text{rows} \ k(B) \) for all \( k \).

In fact, it suffices to assume that \( \text{supp} \ s(A) \supseteq \text{supp} \ s(B) \).

Example. \( A = B = s_A = s_{41} + s_{32} + 2s_{311} + s_{221} + s_{2111} \) \( s_B = s_{41} + 2s_{32} + s_{311} + s_{221} \).

So \( s_A - s_B \) is not Schur-positive but \( \text{supp} \ s(A) \supseteq \text{supp} \ s(B) \).
Necessary conditions for Schur-positivity

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In fact, it suffices to assume that $\text{supp}_s(A) \supseteq \text{supp}_s(B)$.

Example.

$$A = \begin{array}{c|c|c|c|c} & & & & \\
& & & & \\
& & & & \\
& & & & \\
\end{array} \quad B = \begin{array}{c|c|c|c|c} & & & & \\
& & & & \\
& & & & \\
& & & & \\
\end{array}$$

$$s_A = s_{41} + s_{32} + 2s_{311} + s_{221} + s_{2111}$$  

$$s_B = s_{41} + 2s_{32} + s_{311} + s_{221}$$  

So $s_A - s_B$ is not Schur-positive but $\text{supp}_s(A) \supseteq \text{supp}_s(B)$.  

Comparing skew Schur functions quasisymmetically

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Equivalent to row overlap conditions

Let $\text{rects}_{k,\ell}(A)$ denote the number of $k \times \ell$ rectangular subdiagrams contained inside $A$.

$$A = \begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \\
\cdot & \cdot & \\
\cdot & \\
\end{array}$$

$\text{rects}_{3,1}(A) = 2$, $\text{rects}_{2,2}(A) = 3$, etc.

Theorem [RSvW]. Let $A$ and $B$ be skew shapes. TFAE:

- $\text{rows}_k(A) = \text{rows}_k(B)$ for all $k$;
- $\text{cols}_\ell(A) = \text{cols}_\ell(B)$ for all $\ell$;
- $\text{rects}_{k,\ell}(A) = \text{rects}_{k,\ell}(B)$ for all $k, \ell$. 

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Summary so far

$s_A - s_B$ is Schur-pos. $\implies$ \text{supp}_s(A) $\supseteq$ \text{supp}_s(B) $\implies$

\begin{align*}
\text{rows}_k(A) &\preceq \text{rows}_k(B) \ \forall k \\
\text{cols}_\ell(A) &\preceq \text{cols}_\ell(B) \ \forall \ell \\
\text{rects}_{k,\ell}(A) &\leq \text{rects}_{k,\ell}(B) \ \forall k, \ell
\end{align*}

Converse is very false.

Example.

$A = B = s_{31} + s_{211}$

New Goal:

Find weaker algebraic conditions on $A$ and $B$ that imply the overlap conditions.

What algebraic conditions are being encapsulated by the overlap conditions?
Summary so far

$s_A - s_B$ is Schur-pos. $\Rightarrow$ \(\text{supp}_s(A) \supseteq \text{supp}_s(B)\) $\Rightarrow$

\[
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\end{align*}
\]

Converse is very false.

Example.

\[
\begin{align*}
A &= \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot
\end{array} \\
S_A &= S_{31} + S_{211} \\
B &= \begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot
\end{array} \\
S_B &= S_{22}
\end{align*}
\]
Summary so far

\[ s_A - s_B \text{ is Schur-pos.} \quad \Rightarrow \quad \text{supp}_s(A) \supseteq \text{supp}_s(B) \quad \Rightarrow \]

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\end{align*} \]

Converse is very false.

Example.

\[ A = \begin{array}{c}
\Box \\
\Box \\
\Box \\
\Box \\
\end{array} \quad B = \begin{array}{c}
\Box \\
\Box \\
\Box \\
\Box \\
\end{array} \]

\[ s_A = s_{31} + s_{211} \quad s_B = s_{22} \]

**New Goal:** Find weaker algebraic conditions on \( A \) and \( B \) that imply the overlap conditions.

**What algebraic conditions are being encapsulated by the overlap conditions?**
$F$-basis of quasisymmetric functions

- Skew shape $A$.
- SYT $T$ of $A$.

Descent set $S(T) = \{3, 5\}$.

Descent composition $\text{comp}(T) = 323$.

Then $s_A$ expands in the basis of fundamental quasisymmetric functions as $s_A = \sum_{T} F_{\text{comp}(T)}$.

Example. $s_{4431/31} = F_{323} + \cdots$.

Facts.

- The $F$ form a basis for the quasisymmetric functions.
- So notions of $F$-positivity and $F$-support make sense.
- Schur-positivity implies $F$-positivity.
- $\text{supp} s(A) \supseteq \text{supp} s(B)$ implies $\text{supp} F(A) \supseteq \text{supp} F(B)$.
Skew shape $A$.

SYT $T$ of $A$.

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Facts.

- The $F$ form a basis for the quasisymmetric functions.
- So notions of $F$-positivity and $F$-support make sense.
- Schur-positivity implies $F$-positivity.
- $\text{supp} s(A) \supseteq \text{supp} s(B) \implies \text{supp} F(A) \supseteq \text{supp} F(B)$.
Then \( s_A \) expands in the basis of **fundamental quasisymmetric functions** as

\[
 s_A = \sum_{\text{SYT } T} F_{\text{comp}(T)}.
\]

**Example.**

\[
 s_{4431/31} = F_{323} + \cdots.
\]
F-basis of quasisymmetric functions

- Skew shape $A$.
- SYT $T$ of $A$.
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Then $s_A$ expands in the basis of fundamental quasisymmetric functions as

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Example.

$$s_{4431/31} = F_{323} + \cdots.$$

Facts.

- The $F$ form a basis for the quasisymmetric functions.
- So notions of $F$-positivity and $F$-support make sense.
- Schur-positivity implies $F$-positivity.
- $\supp_s(A) \supseteq \supp_s(B)$ implies $\supp_F(A) \supseteq \supp_F(B)$
New results: filling the gap

**Theorem.** [McN. (2013)]

\[ s_A - s_B \text{ is Schur-pos.} \quad \Rightarrow \quad \text{supp}_s(A) \supseteq \text{supp}_s(B) \]

\[ s_A - s_B \text{ is } F\text{-positive} \quad \Rightarrow \quad \text{supp}_F(A) \supseteq \text{supp}_F(B) \quad \Rightarrow \quad \]

\[ \text{rows}_k(A) \nleq \text{rows}_k(B) \quad \forall k \]

\[ \text{cols}_\ell(A) \nleq \text{cols}_\ell(B) \quad \forall \ell \]

\[ \text{rects}_{k,\ell}(A) \leq \text{rects}_{k,\ell}(B) \quad \forall k, \ell \]

Conjecture. The rightmost implication is iff.

Evidence. Conjecture is true for:

- \( n \leq 12; \)
- horizontal strips;
- \( F\)-multiplicity-free skew shapes (as determined by Christine Bessenrodt and Steph van Willigenburg (2013));
- ribbons whose rows all have length at least 2.
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\[ \Rightarrow \quad \text{rows}_k(A) \preccurlyeq \text{rows}_k(B) \quad \forall k \]

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\[ \text{rows}_k(A) \preceq \text{rows}_k(B) \ \forall k \]

\[ \text{cols}_\ell(A) \preceq \text{cols}_\ell(B) \ \forall \ell \]

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$n = 6$ example

$F$-support containment

Row overlap reverse dominance
$n = 12$ case has 12,042 edges.
Adding other bases

\[ s_A - s_B \text{ is } D\text{-positive} \]
\[ \Downarrow \]
\[ s_A - s_B \text{ is Schur-} \text{positive} \]
\[ \Downarrow \]
\[ s_A - s_B \text{ is } S\text{-positive} \]
\[ \Downarrow \]
\[ s_A - s_B \text{ is } F\text{-positive} \]
\[ \Downarrow \]
\[ s_A - s_B \text{ is } M\text{-positive} \]

\[ \supp_D(A) \supseteq \supp_D(B) \]
\[ \supp_s(A) \supseteq \supp_s(B) \]
\[ \supp_S(A) \supseteq \supp_S(B) \]
\[ \supp_F(A) \supseteq \supp_F(B) \]
\[ \supp_M(A) \supseteq \supp_M(B) \]

\[ \Rightarrow \]

\[ \Rightarrow \]

\[ \Rightarrow \]

\[ \Rightarrow \]

\[ \text{rows}_k(A) \preceq \text{rows}_k(B) \forall k \]
\[ \text{cols}_\ell(A) \preceq \text{cols}_\ell(B) \forall \ell \]
\[ \text{rects}_{k,\ell}(A) \leq \text{rects}_{k,\ell}(B) \forall k, \ell \]
Adding other bases

\[ s_A - s_B \text{ is } D\text{-positive} \]
\[ \Rightarrow \]
\[ s_A - s_B \text{ is } \text{Schur-} \text{pos.} \]
\[ s_A - s_B \text{ is } S\text{-positive} \]
\[ \Rightarrow \]
\[ s_A - s_B \text{ is } F\text{-positive} \]
\[ \Rightarrow \]
\[ s_A - s_B \text{ is } M\text{-positive} \]
\[ \Rightarrow \]

\[ \supp_D(A) \supseteq \supp_D(B) \]
\[ \supp_s(A) \supseteq \supp_s(B) \]
\[ \supp_S(A) \supseteq \supp_S(B) \]
\[ \Rightarrow \]
\[ \supp_F(A) \supseteq \supp_F(B) \]
\[ \Rightarrow \]
\[ \supp_M(A) \supseteq \supp_M(B) \]

\[ \begin{align*}
\text{rows}_k(A) & \preccurlyeq \text{rows}_k(B) \forall k \\
\text{cols}_\ell(A) & \preccurlyeq \text{cols}_\ell(B) \forall \ell \\
\text{rects}_{k,\ell}(A) & \leq \text{rects}_{k,\ell}(B) \forall k, \ell
\end{align*} \]