Background & Motivation

Pattern order: order permutations by pattern containment
e.g., $4132 \leq 516423$

Example of a PPP interval

Motivating question [Wilf]

What's the Möbius function of such intervals?

Definitions.

Direct sum: $21 \oplus 3214 = 215436$.

$\pi$ is layered if it takes the form
$\pi = \pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_k$ with each $\pi_i$ decreasing.
e.g. $21 \oplus 321 \oplus 321 \oplus 1 = 215438769$

Main Results

Fact [Björner]:
Non-trivial disconnected subinterval $\implies$ not shellable

Theorem (almost all intervals are not shellable)

Fix $\sigma$. Randomly choose $\tau$ of length $n$.

$$\lim_{n \to \infty} \text{Probability that } \Delta(\sigma, \tau) \text{ is shellable} = 0.$$ 

In contrast, there's a large class of intervals that are shellable.

Open Problems

Understand non-shellable intervals without disconnected subintervals.
e.g. $[123, 3416725]$.

Find a good way to test shellability by computer.

Separable permutations: can be built from 1 by a sequence of direct sums or skew sums.

$1 \oplus 1 = 12$

$(1 \oplus 1) \oplus (1 \oplus 1) = 12 \oplus 12 = 3412$

$1 \oplus 3412 = 14523$ etc.

Conjecture. Suppose $\sigma, \tau$ separable such that $[\sigma, \tau]$ does not contain a non-trivial disconnected subinterval. Then $[\sigma, \tau]$ is shellable.

Conjecture. $[\sigma, \tau]$ is always rank unimodal.

Take-home idea

The PPP is hard! But there are substantial classes where results have been pried out in recent years, and almost certainly more is to come.

Supported by the Simons Foundation (McNamara) and the Icelandic Research Fund (Steingrímsson)