Cylindric Skew Schur Functions

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Schur functions

- Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$.

- Example: $(4, 4, 3, 1)$
Schur functions

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- Example: $(4, 4, 3, 1)$
- Semistandard Young tableau (SSYT)

Schur function $s_\lambda$ in the variables $x = (x_1, x_2, \ldots)$ defined by

$$s_\lambda(x) = \sum_{SSYT} x^T = \sum_{SSYT} x_{#1's in T}^1 x_{#2's in T}^2 \cdots$$

$$s_{4431}(x) = x_1 x_3^2 x_4^4 x_5^2 x_6^2 x_7 x_9 + \cdots.$$
**Skew Schur functions**

- Partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$.
- $\mu$ fits inside $\lambda$: form $\lambda/\mu$.
- Example: $(4, 4, 3, 1)/(3, 1)$
- Semistandard Young tableau (SSYT)

**Skew Schur function** $s_{\lambda/\mu}$ in the variables $x = (x_1, x_2, \ldots)$ defined by

$$s_{\lambda/\mu}(x) = \sum_{SSYT \ T} x^T = \sum_{SSYT \ T} x_1^\# \text{1's in } T \cdot x_2^\# \text{2's in } T \cdot \ldots$$

$$s_{4431}(x) = x_1 x_3^2 x_4^4 x_5 x_6^2 x_7 x_9 + \ldots$$
Schur functions are symmetric functions
Schur functions $s_\lambda$ form a basis for the symmetric functions.
Arise in: representation theory of the symmetric group $S_n$.
They are the characters of the irreducible representations of $GL(n, \mathbb{C})$.
Correspond to Schubert classes in $H^*(Gr_{kn})$. 
For skew Schur?

- Skew Schur functions are symmetric functions

\[ s_{\lambda/\mu}(x) = \sum_{\nu} c_{\lambda\mu}^{\nu} s_{\nu}(x). \]

- \( c_{\lambda\mu}^{\nu} \): Littlewood-Richardson coefficients

- Since \( c_{\lambda\mu}^{\nu} \geq 0 \), they are Schur-positive.

\[ s_{4431/31} = s_{44} + 2s_{431} + s_{422} + s_{4211} + s_{332} + s_{3311}. \]

- Schur-positive symmetric functions are significant in the representation theory of \( S_n \).
Cylindric skew Schur functions

- Infinite skew shape $C$
- Invariant under translation
- Identify $(x, y)$ and $(x - n + k, y + k)$.
Cylindric skew Schur functions

- Infinite skew shape $C$
- Invariant under translation
- Identify $(x, y)$ and $(x - n + k, y + k)$.
- Entries weakly increasing in each row
- Strictly increasing up each column
- Alternatively: SSYT with relations between entries in first and last columns

\[ s_C(x) = \sum_{T} x^T = \sum_{T} x_1^{\#1\text{'s in } T} x_2^{\#2\text{'s in } T} \ldots. \]

- $s_C$ is a symmetric function
Cylindric skew Schur functions

**Example**

- Gessel, Krattenthaler: “Cylindric Partitions”
- Bertram, Ciocan-Fontanine, Fulton: “Quantum Multiplication of Schur Polynomials”
- Postnikov: “Affine Approach to Quantum Schubert Calculus” math.CO/0205165
- Stanley: “Recent Developments in Algebraic Combinatorics” math.CO/0211114
Motivation 1: $P$-partitions and an old conjecture of Stanley
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$P$: partially ordered set (poset)

$\omega : P \rightarrow \{1, 2, \ldots, |P|\}$

bijective labelling

**Definition** (R. Stanley) Given a labelled poset $(P, \omega)$, a $(P, \omega)$-partition is a map $f : P \rightarrow \mathbb{P}$ with the following properties:

- $f$ is order-preserving: If $x \leq y$ in $P$ then $f(x) \leq f(y)$
- If $x \prec y$ in $P$ and $\omega(x) > \omega(y)$ then $f(x) < f(y)$
Motivation 1: \( P \)-partitions and an old conjecture of Stanley

\( P \): partially ordered set (poset)
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- $f$ is order-preserving: If $x \leq y$ in $P$ then $f(x) \leq f(y)$
- If $x \lhd y$ in $P$ and $\omega(x) > \omega(y)$ then $f(x) < f(y)$

$$K_{P,\omega}(x) = \sum_{f} x^{f} = \sum_{f} x^{\#f^{-1}(1)} x^{\#f^{-1}(2)} \cdots .$$
A non-symmetric example

\[ K_{P,\omega}(x) = \sum_{f} x^T = \sum_{f} f^{-1}(1) f^{-1}(2) \ldots . \]

**Example**

Coefficient of \( x_1^2 x_2 x_3 = 1 \)
Coefficient of \( x_1 x_2^2 x_3 = 0 \)
\[ \Rightarrow \text{not symmetric} \]
**Schur labelled skew shape posets and Stanley’s \( P \)-partitions Conjecture**

Bijection: SSYT of shape \( \lambda/\mu \) ↔ \( (P, \omega) \)-partitions

Furthermore,

\[ K_{P,\omega}(x) = s_{\lambda/\mu}(x). \]

**BIG QUESTION** What other labelled posets \( (P, \omega) \) have symmetric \( K_{P,\omega}(x) \) ?
**Schur labelled skew shape posets and Stanley’s $P$-partitions Conjecture**

Bijection: SSYT of shape $\lambda/\mu \leftrightarrow (P, \omega)$-partitions

Furthermore,

$$K_{P,\omega}(x) = s_{\lambda/\mu}(x).$$

**BIG QUESTION** What other labelled posets $(P, \omega)$ have symmetric $K_{P,\omega}(x)$?

**Conjecture** (Stanley, c.1971) $K_{P,\omega}(x)$ is symmetric if and only if $(P, \omega)$ is isomorphic to a (Schur labelled) skew shape poset.
Connection to cylindric skew Schur functions

**Example**

We can check that \( K_{P,\omega}(x) \) is symmetric.
So does it obey Stanley’s conjecture?
**Example**

We can check that $K_{P, \omega}(x)$ is symmetric. So does it obey Stanley’s conjecture?
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So does it obey Stanley’s conjecture?

\[ \omega(a) > \omega(c) > \omega(b) > \omega(d) > \omega(a) \]

Yikes!
**Example**

We can check that $K_{P,\omega}(x)$ is symmetric. So does it obey Stanley’s conjecture?

$\omega(a) > \omega(c) > \omega(b) > \omega(d) > \omega(a)$ Yikes! Oriented Poset
\((P, O)\)-partitions

Labelled poset \((P, \omega)\)

\[ K_{P,\omega}(x) \]

skew shape posets
skew Schur functions

Oriented poset \((P, O)\)

\[ K_{P,O}(x) \]

cylindric skew shape posets
cylindric skew Schur functions
**Malvenuto’s reformulation**

**Theorem** (C. Malvenuto, c. 1995) A labelled poset is a skew shape poset if and only if every connected component has no forbidden convex subposets.

**Theorem** (McN.) An oriented poset is a cylindric skew shape poset if and only if every connected component has no forbidden convex subposets.

**Conjecture** (Stanley) $K_{P,\omega}(x)$ is symmetric if and only if every connected component of $(P, \omega)$ is isomorphic to a skew shape poset.

**Conjecture** (Stanley’s conjecture extended to oriented posets) $K_{P,O}(x)$ is symmetric if and only if every connected component of $(P, O)$ is isomorphic to a cylindric skew shape poset.
Extended version is false!
**Motivation 2: Positivity of Gromov-Witten invariants**

In $H^*(Gr_{kn})$,

$$\sigma_\lambda \sigma_\mu = \sum_{\nu \subseteq k \times (n-k)} c^\nu_{\lambda\mu} \sigma_\nu.$$ 

In $QH^*(Gr_{kn})$,

$$\sigma_\lambda \ast \sigma_\mu = \sum_{d \geq 0} \sum_{\nu \vdash |\lambda| + |\mu| - dn} q^d C_{\lambda\mu}^{\nu,d} \sigma_\nu.$$ 

$C_{\lambda\mu}^{\nu,d} = 3$-point Gromov-Witten invariants

$= \# \{ \text{rational curves of degree } d \text{ in } Gr_{kn} \text{ that meet fixed generic translates of the Schubert varieties } \Omega_{\lambda\lor}, \Omega_\lambda \text{ and } \Omega_\mu \}.$

**Key point:** $C_{\lambda\mu}^{\nu,d} \geq 0$.

“Fundamental Open Problem”: 
Motivation 2: Positivity of Gromov-Witten invariants

In $H^*(Gr_{kn})$, 

$$\sigma_\lambda \sigma_\mu = \sum_{\nu \subseteq k \times (n-k)} C_{\lambda \mu}^\nu \sigma_\nu.$$ 

In $QH^*(Gr_{kn})$, 

$$\sigma_\lambda \ast \sigma_\mu = \sum_{d \geq 0} \sum_{\nu \vdash |\lambda|+|\mu|-dn} q^d C_{\lambda \mu}^{\nu,d} \sigma_\nu.$$ 

$C_{\lambda \mu}^{\nu,d} = 3$-point Gromov-Witten invariants 

$= \# \{ \text{rational curves of degree } d \text{ in } Gr_{kn} \text{ that meet fixed generic translates of the Schubert varieties } \Omega_{\nu}, \Omega_{\lambda} \text{ and } \Omega_{\mu} \}$. 

Key point: $C_{\lambda \mu}^{\nu,d} \geq 0$. 

“Fundamental Open Problem”: Find an algebraic or combinatorial proof of this fact.
What’s cylindric got to do with it?

**Theorem (Postnikov)**

\[
s_{\lambda/d/\mu}(x_1, \ldots, x_k) = \sum_{\nu \subseteq k \times (n-k)} C_{\lambda\mu}^{\nu,d} s_{\nu}(x_1, \ldots, x_k).
\]

**Conclusion:** Want to understand expansions of cylindric skew Schur functions into Schur functions.
**Theorem** (Postnikov)

\[ s_{\lambda/d/\mu}(x_1, \ldots, x_k) = \sum_{\nu \subseteq k \times (n-k)} C_{\lambda/\mu}^{\nu,d} s_\nu(x_1, \ldots, x_k). \]

**Conclusion:** Want to understand expansions of cylindric skew Schur functions into Schur functions.

**Corollary** $s_{\lambda/d/\mu}(x_1, x_2, \ldots, x_k)$ is Schur-positive.

**Known:** $s_{\lambda/d/\mu}(x_1, x_2, \ldots)$ need not be Schur-negative.

**Note:** $s_{\lambda}(x_1, x_2, \ldots, x_k) \neq 0 \iff \lambda$ has at most $k$ rows.

**Example:** If $s_{\lambda/d/\mu} = s_{22} + s_{211} - s_{1111}$, then $s_{\lambda/d/\mu}(x_1, x_2, x_3) = s_{22} + s_{211}$ is Schur-positive.
When is a cylindric skew Schur function Schur-positive?

**Theorem (McN.)** For any cylindric skew shape \( C \),

\[ s_C(x_1, x_2, \ldots) \text{ is Schur-positive} \iff C \text{ is a skew shape}. \]

Equivalently, if \( C \) is a non-trivial cylindric skew shape, then \( s_C(x_1, x_2, \ldots) \) is not Schur-positive.
Example: Cylindric ribbons

Example

\[ C : \]

\[ s_C(x_1, x_2, \ldots) = \sum_{\nu \subseteq k \times (n-k)} c_{\nu} s_{\nu} + s_{n-k,1}^k - s_{n-k-1,1}^{k+1} + s_{n-k-2,1}^{k+2} - \cdots + (-1)^{n-k} s_{1n}. \]
**Example: Cylindric ribbons**

**Example**

\[ C: \]

\[
 s_C(x_1, x_2, \ldots) = \sum_{\nu \subseteq k \times (n-k)} c_{\nu} s_{\nu} + s_{n-k,1^k} - s_{n-k-1,1^{k+1}} + s_{n-k-2,1^{k+2}} - \cdots + (-1)^{n-k} s_{1^n}. 
\]

Schur-positive with \( k + 1 \) variables

**Not** Schur-positive with \( \geq k + 2 \) variables

**General cylindric skew shape:** \( \geq k + 2 + l \) variables

**Shapes in Postnikov's theorem:** \( \geq 2k + 1 \) variables
Tool: Cylindric skew Schur functions as signed sums of skew Schurs

Bertram, Ciocan-Fontanine, Fulton:

😊 Nice description in terms of ribbons
😊 Only for certain shapes, certain terms

Gessel, Krattenthaler:

😊 Works for all cylindric skew shapes
😊 Not as nice a description

We can get the best of both worlds:
A technique for expanding a cylindric skew Schur function in terms of skew Schur functions that
Works for all cylindric skew shapes like G-K and
has a nice description like B-CF-F
**Theorem (B-CF-F)** For $\lambda$, $\mu$, $\nu \subseteq k \times (n - k)$ with $|\mu| + |\nu| = |\lambda| + dn$ for some $d \geq 0$, we have

$$C_{\mu \nu}^{\lambda, d} = \sum_{\tau} \varepsilon(\tau/\lambda) c_{\mu \nu}^{\tau}$$

where the sum is over all $\tau$ with $\tau_1 \leq n - k$ that can be obtained from $\lambda$ by adding $d$ $n$-ribbons.
**Theorem (B-CF-F)** For $\lambda, \mu, \nu \subseteq k \times (n - k)$ with $|\mu| + |\nu| = |\lambda| + dn$ for some $d \geq 0$, we have

$$
\sum_{\nu} C_{\mu\nu}^{\lambda,d} s_{\nu}(x_1, \ldots, x_k) = \sum_{\nu} \sum_{\tau} \varepsilon(\tau / \lambda) c_{\mu\nu}^{\tau} s_{\nu}(x_1, \ldots, x_k)
$$

where the sum is over all $\tau$ with $\tau_1 \leq n - k$ that can be obtained from $\lambda$ by adding $dn$-ribbons.
Formula of
Bertram, Ciocan-Fontanine, Fulton

**Theorem (B-CF-F)** For \( \lambda, \mu, \nu \subseteq k \times (n - k) \) with 
\[ |\mu| + |\nu| = |\lambda| + dn \]
for some \( d \geq 0 \), we have

\[
\sum_{\nu} C_{\mu\nu}^{\lambda,d} s_{\nu}(x_1, \ldots, x_k) = \sum_{\nu} \sum_{\tau} \varepsilon(\tau / \lambda) c_{\mu\nu}^{\tau} s_{\nu}(x_1, \ldots, x_k)
\]

where the sum is over all \( \tau \) with \( \tau_1 \leq n - k \) that can be obtained from \( \lambda \) by adding \( d n \)-ribbons.

**Corollary** For any cylindric skew shape \( \lambda/d/\mu \) with 
\( \lambda, \mu \subseteq k \times (n - k) \), we have

\[
s_{\lambda/d/\mu}(x_1, \ldots, x_k) = \sum_{\tau} \varepsilon(\tau / \lambda) s_{\tau/\mu}(x_1, \ldots, x_k),
\]

where the sum is over all \( \tau \) with \( \tau_1 \leq n - k \) that can be obtained from \( \lambda \) by adding \( d n \)-ribbons.
Tool: Cylindric skew Schur functions as signed sums of skew Schurs

Example
Tool: Cylindric skew Schur functions as signed sums of skew Schurs

Example

\[ k \]

\[ n-k \]
EXAMPLE

\[ \frac{n-k}{k} \]
Tool: Cylindric skew Schur functions as signed sums of skew Schurs

Example

\[
\begin{array}{c}
\text{k} \\
\hline
\text{n-k}
\end{array}
\]

= 

[Diagram of green and red block structures]
Tool: Cylindric skew Schur functions as signed sums of skew Schurs

Example
Tool: Cylindric skew Schur functions as signed sums of skew Schurs

Example
**Tool: Cylindric skew Schur functions as signed sums of skew Schurs**

**EXAMPLE**

\[
SC = s_{333211}/21 - s_{3322111}/21 + s_{331111111}/21
= s_{3331} + s_{3322} + s_{33211} + s_{322111} + s_{31111111}
- s_{222211} - s_{2221111} + s_{22111111} + s_{211111111}.
\]
Consequence: Lots of skew Schur function identities
Example: Cylindric ribbons

\[ C: \]

\[
SC(x_1, x_2, \ldots) = \sum_{\nu \subseteq k \times (n-k)} c_{\nu} s_{\nu} + s_{n-k,1^k} - s_{n-k-1,1^{k+1}} + s_{n-k-2,1^{k+2}} - \cdots + (-1)^{n-k} s_{1^n}.
\]
**Example: Cylindric ribbons**

\[ s_C(x_1, x_2, \ldots) = \sum_{\nu \subseteq k \times (n-k)} c_{\nu}s_{\nu} + s_{n-k,1^k} - s_{n-k-1,1^{k+1}} \]

\[ + s_{n-k-2,1^{k+2}} - \cdots + (-1)^{n-k}s_{1^n}. \]

However, \[ s_C(x_1, x_2, \ldots) = \sum_{\nu \subseteq k \times (n-k)} c_{\nu}s_{\nu} + s_H. \]

\( s_C \): cylindric skew Schur function

\( s_H \): cylindric Schur function

We say that \( s_C \) is **cylindric Schur-positive**.
Conjecture

**Conjecture** For any cylindric skew shape $C$, $s_C$ is cylindric Schur-positive.
**Conjecture**  For any cylindric skew shape \( C \), \( s_C \) is cylindric Schur-positive.

**Theorem**  (McN.) The cylindric Schur functions corresponding to a given translation \((−n+k, +k)\) are linearly independent.

**Theorem**  (McN.) If \( s_C \) can be written as a linear combination of cylindric Schur functions with the same translation as \( C \), then \( s_C \) is cylindric Schur-positive.