A CLASS OF TRIDIAGONAL REPRODUCING KERNELS

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Abstract. The class of analytic reproducing kernels

\[ K_p(z, w) = \sum_{n=0}^{\infty} f_n(z) \overline{f_n(w)} \]

is considered where \( f_n(z) = (1 - b_n z) z^n \) with \( b_n = \left( \frac{n+1}{n+2} \right)^p \) and \( p > 0 \). In this case \( H(K_p) \) consists of functions with domain \( \mathbb{D} \cup \{1\} \). For each \( p \), a concrete realization of \( H(K_p) \) is provided. For the case \( p > 1/2 \), \( H(K_p) \) is shown to have the factorization property and the operator of multiplication by \( z \) is shown to be similar to a rank one perturbation of the unilateral shift. A characterization of the multiplier algebra of \( H(K_p) \) is given for all values of \( p > 0 \).

1. Introduction

The function \( K(z, w) \) is positive definite (denoted \( K \gg 0 \)) on the set \( E \times E \) if for any finite collection \( z_1, z_2, \ldots, z_n \) in \( E \subseteq \mathbb{C} \) and any complex numbers \( \alpha_1, \alpha_2, \ldots, \alpha_n \), the sum \( \sum_{i,j=1}^{n} \bar{\alpha}_i \alpha_j K(z_i, z_j) \) is non-negative. It is well known that if \( K \gg 0 \) on \( E \), then the set of functions in \( z \) given by

\[
\left\{ \sum_{j=1}^{n} \alpha_j K(z, w_j) : \alpha_1, \ldots, \alpha_n \in \mathbb{C}, w_1, \ldots, w_n \in E \right\}
\]

has dense span in a Hilbert space \( H(K) \) of functions on \( E \) with

\[
|| \sum_{j=1}^{n} \alpha_j K(z, w_j) ||^2 = \sum_{i,j=1}^{n} \bar{\alpha}_i \alpha_j K(w_i, w_j).
\]
A fundamental property of $H(K)$ is the \textit{Reproducing Property} which states that $f(w) = \langle f(z), K(z, w) \rangle$ for every $w$ in $E$ and $f$ in $H(K)$. Thus evaluation at $w$ is a bounded linear functional for each $w$ in $E$.

Conversely, it is well known that if $F$ is a Hilbert space of functions defined on $E$ such that evaluation at $w$ is a bounded linear functional for each $w$ in $E$, then there is a unique $K$ defined on $E \times E$ such that $F = H(K)$. It follows from the reproducing property that $K(z, w) = \overline{K(w, z)}$. Hence if $K$ is analytic in the first variable, then $K$ is coanalytic in the second variable. In this case $K$ is an \textit{analytic} kernel. It is well known, see Adams, McGuire, and Paulsen [2], that if $K$ is an analytic kernel with series expansion $K(z, w) = \sum_{i,j=0}^{\infty} a_{i,j} z^i \overline{w}^j$ about $(0,0)$ and $A = [a_{i,j}]$ is factored as $A = BB^*$, then $H(K)$ is identifiable with the range space of $B$ in $l^2$. Recall the range space of $B$ is given by $\mathcal{R}(B) = \{ B\vec{x} : \vec{x} \in l^2 \}$ with $||B\vec{x}||_{\mathcal{R}(B)} = ||\vec{x}||_{l^2}$. The column vectors $\{b_j\}_{j=0}^{\infty}$ of $B$ given by $\vec{b}_j = \begin{pmatrix} b_{0,j} \\ b_{1,j} \\ b_{2,j} \\ \vdots \end{pmatrix}$ correspond to an orthonormal basis $\{f_j(z)\}_{j=0}^{\infty}$ of $H(K)$ where $f_j(z) = \sum_{i=0}^{\infty} b_{i,j} z^i$. An important observation is that if $K_1$ and $K_2$ are two such analytic kernels with associated factorizations $A_1 = B_1 B_1^*$ and $A_2 = B_2 B_2^*$, then $H(K_1) \subset H(K_2)$ if and only if the range of $B_1$ is contained in the range of $B_2$.

In Shields [12], multiplication operators on analytic reproducing kernel Hilbert spaces with kernels of the form $K(z, w) = \sum_{n=0}^{\infty} a_n z^n \overline{w}^n$ were extensively studied. In these spaces the monomials $\{\sqrt{a_n} z^n\}$ form an orthonormal basis, and the operator $M_z$ of multiplication by $z$ is a forward unilateral shift. Richter [11] extended the work of Shields [12] to study the invariant subspace structure of multiplication by $z$ on certain Banach spaces, $\mathcal{B}$, of analytic functions in which evaluation is continuous and for which the
following Factorization Property holds: if \( f \in \mathcal{B} \) and \( f(\lambda) = 0 \), then there exists \( g \in \mathcal{B} \) such that \((z - \lambda)g = f\).

In Adams and McGuire [1], a study was begun of the spaces with kernels of the form \( K(z, w) = \sum_{n=0}^{\infty} f_n(z)f_n(w) \) where \( f_n(z) = (a_{n,0} + a_{n,1}z + \cdots + a_{n,J}z^J)z^n \) and \( J \) is fixed. These spaces are known as bandwidth \( J \) spaces since the Taylor series expansion of \( K(z, w) = \sum_{i,j=0}^{\infty} a_{i,j}z^i\bar{w}^j \) satisfies \( a_{i,j} = 0 \) outside the band \(|i - j| \leq J\). In this case, the polynomials \( \{f_n(z)\} \) form an orthonormal basis for \( H(K) \). It was shown in Adams and McGuire [1] that the behavior of the multiplication operators on these spaces can be markedly different from the Shields case \((J = 0)\).

This paper is focused on a special class of tridiagonal kernels \((J = 1)\) that bring this difference into a sharper focus. The class is defined for each \( p > 0 \) and \( n \in \mathbb{N} \) by setting \( f_n(z) = (1 - b_n z)z^n \) where \( b_n = (n+1)/(n+2) \), resulting in the kernel \( K_p(z, w) = \sum_{n=0}^{\infty} f_n(z)f_n(w) \). It is straightforward to verify that the domain of \( K_p \) is given by \( \mathcal{D}(K_p) = \{(z, w) \in \mathbb{D} \cup \{1\}\} \), that \( \{b_n\} \) is a sequence of positive numbers that increases to 1, and that \( K_p(z, 1) = \sum_{n=0}^{\infty} (1 - b_n)(1 - b_n z)z^n \).

The principle result of this paper is a functional decomposition of the space \( H(K_p) \) for \( p > 0 \). This decomposition allows us to determine that the operator \( M_z \) is bounded if and only if \( p > \frac{1}{2} \) and, in this case, to completely characterize the multiplier algebra of \( H(K_p) \). For \( 0 < p < \frac{1}{2} \), we provide necessary and sufficient conditions for a function \( \phi \) to be a multiplier of \( H(K_p) \). Additionally, we show that for \( p > \frac{1}{2} \), \( H(K_p) \) satisfies the factorization property of Richter [11]. From this it easily follows from [11] that \( M_z^* \) is in the Cowen-Douglas class \( B_1 \), \( M_z \) is a cellular indecomposable operator, and that the invariant subspaces of \( M_z \) are either of the form \((1 - z)M \) where \( M = \psi H^2 \) for some inner function \( \psi \) or the span of \((1 - z)M \) and the function \( K_p(z, 1) \).
2. Main Results

Our first result shows that the functions in $H(K_p)$ can be decomposed into $(1-z)$ times an $H^2$ function plus a scalar multiple of the function $K_p(z,1)$. Our second and more difficult result determines precisely which $H^2$ functions can occur in the factorization and the dependency on $p$. Before proceeding, we include without proof a lemma that contains a few obvious facts that will be useful in the proofs of these results.

Lemma 2.1. Let $\mathcal{P}$ denote the collection of matrices with non-negative components, let $A \ast B$ denote the Schur or Hadamard product of the matrices $A$ and $B$, and let $V_+$ denote the collection of unit vectors in $l^2_+$ whose components are non-negative.

1. If $A \in \mathcal{P}$, then $||A|| = \sup_{\vec{v} \in V_+} ||A\vec{v}||$.
2. If $A_1, A_2 \in \mathcal{P}$, then $||A_1|| \leq ||A_1 + A_2|| \leq ||A_1|| + ||A_2||$.
3. If $A, B \in \mathcal{P}$ with $B = [b_{j,k}]$, and $0 < \lambda \leq b_{j,k} \leq \gamma < \infty$ for each $j, k$, then $\lambda||A|| \leq ||A \ast B|| \leq \gamma||A||$.

Theorem 2.2. If $f \in H(K_p)$, then $f(z) = (1-z)g(z) + \alpha K_p(z,1)$ for some $g$ in the Hardy space $H^2(\mathbb{D})$ and $\alpha \in \mathbb{C}$.

Proof. First note that if $f \in H(K_p)$ and $Q$ is the projection of $H(K_p)$ onto the one dimensional span of $K_p(z,1)$, then $f = (I - Q)f + Qf$. Since

$$Qf = \langle f, \frac{K_p(z,1)}{\sqrt{K_p(1,1)}} \rangle \frac{K_p(z,1)}{\sqrt{K_p(1,1)}} = \frac{f(1)}{K_p(1,1)} K_p(z,1),$$

$(Qf)(1) = \langle Qf, K_p(z,1) \rangle = f(1) = \langle f, K_p(z,1) \rangle$ and $(Qf)(1) = \langle (I - Q)f, K_p(z,1) \rangle = 0$. Thus it suffices to show that if $f \in H(K_p)$ and $f(1) = \langle f, K_p(z,1) \rangle = 0$, then $f(z) = (1-z)g(z)$ for some $g \in H^2(\mathbb{D})$. Writing $f(z) = \sum_{n=0}^{\infty} a_n f_n(z) = \sum_{n=0}^{\infty} a_n (1-b_n z)z^n$ we note that the condition that $f(1) = 0$ implies $\sum_{n=0}^{\infty} a_n (1-b_n) = 0$. In order that

$$f(z) = (1-z)g(z) = (1-z) \sum_{n=0}^{\infty} g_n z^n = g_0 + \sum_{n=1}^{\infty} (g_n - g_{n-1})z^n$$
for some \( g \in H^2(\mathbb{D}) \) we must produce a sequence \( \{g_n\} \) in \( l^2 \) such that

\[
g_0 + \sum_{n=1}^{\infty} (g_n - g_{n-1}) z^n = \sum_{n=0}^{\infty} \alpha_n (1 - b_n z) z^n = \alpha_0 + \sum_{n=1}^{\infty} (\alpha_n - \alpha_{n-1} b_{n-1}) z^n.
\]

This leads to the recursion \( g_0 = \alpha_0 \) and \( g_n = g_{n-1} + \alpha_n - \alpha_{n-1} b_{n-1} \) for \( n \geq 1 \). Thus

\[
g_1 = g_0 + \alpha_1 - \alpha_0 b_0 = \alpha_0 (1 - b_0) + \alpha_1,
\]

\[
g_2 = g_1 + \alpha_2 - \alpha_1 b_1 = \alpha_0 (1 - b_0) + \alpha_1 (1 - b_1) + \alpha_2,
\]

and the \( n^{th} \) term is given by

\[
g_n = \left( \sum_{k=0}^{n-1} \alpha_k (1 - b_k) \right) + \alpha_n.
\]

Since \( \sum_{k=0}^{\infty} \alpha_k (1 - b_k) = 0 \), for \( n \geq 1 \) the sum

\[
\sum_{k=0}^{n-1} \alpha_k (1 - b_k) = - \sum_{k=n}^{\infty} \alpha_k (1 - b_k) = 0
\]

and hence \( g_n = \alpha_n - \sum_{k=n}^{\infty} \alpha_k (1 - b_k) \). Since \( \{\alpha_n\} \) is an \( l^2 \) sequence, it suffices to show \( \{\sum_{k=n}^{\infty} \alpha_k (1 - b_k)\}_{n=1}^{\infty} \) is an \( l^2 \) sequence. Since

\[
\{\sum_{k=n}^{\infty} \alpha_k (1 - b_k)\}_{n=1}^{\infty} = B_p \{\alpha_n\}_{n=1}^{\infty}
\]

where

\[
B_p = \begin{pmatrix}
1 - b_0 & 1 - b_1 & 1 - b_2 & \cdots \\
0 & 1 - b_1 & 1 - b_2 & \cdots \\
0 & 0 & 1 - b_2 & \cdots \\
\vdots & \vdots & \ddots & \ddots
\end{pmatrix},
\]

it is enough to show that \( B_p \) is a bounded matrix.

The tangent line approximation to \( f(x) = 1 - x^p \) at \( x = 1 \) is given by

\[-p(x - 1).\]

Since \( \lim_{n \to \infty} n+\frac{1}{n+2} = 1 \), for large \( n \), \( 1 - b_n = 1 - \left( \frac{n+1}{n+2} \right)^p \) can be
approximated by \(-p\left(\frac{n+1}{n+2} - 1\right) = \frac{p}{n+2}\). A straightforward application of part (3) of Lemma 2.1 shows that \(B_p\) is bounded if and only if the matrix

\[
C_p = \begin{pmatrix}
\frac{p}{2} & \frac{p}{3} & \frac{p}{4} & \cdots \\
0 & \frac{p}{3} & \frac{p}{4} & \cdots \\
0 & 0 & \frac{p}{4} & \cdots \\
0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

is bounded. It is well known that the Cesaro operator

\[
C = \begin{pmatrix}
1 & 0 & 0 & 0 & \cdots \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \cdots \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

is bounded, see Brown, Halmos, Shields [7]. Let \(Q_0\) denote the projection onto the first canonical basis vector of \(l^2\) and note \(C_p = p(I-Q_0)C^*(I-Q_0)\) is bounded which establishes the result.

Our next result provides a more explicit description of the nature of the decomposition of \(H(K_p)\) that was obtained in Theorem 2.2. For convenience we will denote the diagonal operator with diagonal entries given by the sequence \(\{a_n\}\) by either of \(D[a_1, a_2, a_3, \ldots]\) or \(D[\{a_n\}]\).

**Theorem 2.3.** If \(A_p = \{g \in H^2(\mathbb{D}) : (1-z)g(z) \in H(K_p)\}\), then

1. for \(p > \frac{1}{2}\), \(A_p = H^2(\mathbb{D})\);  
2. for \(p = \frac{1}{2}\), \(A_p\) is dense in \(H^2(\mathbb{D})\), but not equal to \(H^2(\mathbb{D})\);  
3. for \(0 < p < \frac{1}{2}\), \(A_p\) is the orthogonal complement in \(H^2(\mathbb{D})\) of the span of \(\{g_p(z)\}\) where \(g_p(z) = \sum_{n=0}^{\infty} (1-b_n)(n+2)^p z^n\).
Proof. We begin with the case where \( p > \frac{1}{2} \). In order to show that \( A_p = H^2 \), it suffices to show that the range of \( A \) is contained in the range of \( B \) where

\[
A = \begin{pmatrix}
1 & 0 & 0 & \cdots \\
-1 & 1 & 0 & \cdots \\
0 & -1 & 1 & \cdots \\
\vdots & \vdots & \ddots & \ddots
\end{pmatrix}
\quad \text{and} \quad
B = \begin{pmatrix}
1 & 0 & 0 & \cdots \\
-(\frac{1}{2})^p & 1 & 0 & \cdots \\
0 & -(\frac{2}{3})^p & 1 & \cdots \\
\vdots & \vdots & \ddots & \ddots
\end{pmatrix}.
\]

By the Range Inclusion Theorem of Douglas [9], it suffices to show there exists a bounded operator \( R = [r_{i,j}] \) such that \( A = BR \). It is a straightforward computation to show that \( R \) must be lower triangular, \( r_{i,i} = 1 \) for each \( i \), and

\[
r_{i,j} = \left( \frac{j+2}{j+1} \right)^p \left[ 1 - \left( \frac{j+1}{j+2} \right)^p \right], \quad \text{if} \quad j < i.
\]

Let \( \alpha_j = \frac{p}{1 - \left( \frac{j+1}{j+2} \right)^p} = \frac{p}{1 - b_j} \) and note that \( \lim_{j \to \infty} \alpha_j = 1 \). Since the diagonal matrix \( D[\{\alpha_j\}] \) is bounded and invertible the matrix \( M \) is bounded if and only if \( MD[\{\alpha_j\}] \) is bounded. Note that

\[
MD[\{\alpha_j\}] = \begin{pmatrix}
0 & 0 & 0 & 0 & \cdots \\
\frac{p}{2} & 0 & 0 & 0 & \cdots \\
\frac{p}{2} \left( \frac{2}{3} \right)^p & \frac{p}{3} & 0 & 0 & \cdots \\
\frac{p}{2} \left( \frac{2}{3} \right)^p & \frac{p}{3} \left( \frac{3}{4} \right)^p & \frac{p}{4} & 0 & \cdots \\
\frac{p}{2} \left( \frac{2}{3} \right)^p & \frac{p}{3} \left( \frac{3}{4} \right)^p & \frac{p}{4} \left( \frac{4}{5} \right)^p & \frac{p}{5} & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix}.
\]

Let \( L_1 = D[\{n^{p-1}\}]_{n=1}^\infty \), \( L_2 = D[\{1/n\}]_{n=1}^\infty \), and \( D_p = D[\{n^{-p}\}]_{n=1}^\infty \). It is straightforward to verify that \( MD[\{\alpha_n\}] = pD_p(C - L_2)L_1D_p^{-1} \) where
C is the Cesaro matrix. Since $pD_pL_2L_1D_p^{-1}$ is a bounded matrix, it easily follows that $MD[\{\alpha_n\}]$ is bounded if and only if $D_pCD_p^{-1} =$

$$
\begin{pmatrix}
1^{1-p} & 0 & 0 & \cdots \\
0 & 2^{1-p} & 0 & \cdots \\
0 & 0 & 3^{1-p} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & \cdots \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\begin{pmatrix}
1^{p-1} & 0 & 0 & \cdots \\
0 & 2^{p-1} & 0 & \cdots \\
0 & 0 & 3^{p-1} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$

is bounded. Our next goal is to show that $D_pCD_p^{-1}$ is bounded if and only if $p > \frac{1}{2}$.

By applying item (3) of Lemma 2.1, the boundedness of $D_pCD_p^{-1}$ can be shown to be equivalent to the boundedness of $E_pC_1E_p^{-1}$ where $E_p = D[1^{1-p}, 2^{1-p}, 4^{1-p}, \ldots, 2^{1-p}, 8^{1-p}, \ldots, 8^{1-p}, \ldots]$ and $C_1$ is the lower triangular matrix

$$
C_1 = 
\begin{pmatrix}
1 & 0 & 0 & \cdots \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$

whose $i,j$th entry is $2^{-k}$ provided $2^k \leq i < 2^{k+1}$ and $0 \leq j \leq i$ for $k = 0, 1, 2, \ldots$.

By applying item (2) of Lemma 2.1 we can augment $C_1$ to obtain the equivalent problem of the boundedness of the matrix $E_pC_2E_p^{-1}$ where $E_p =$
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$D[1^{-p}, 21^{-p}, 21^{-p}, 41^{-p}, \ldots, 41^{-p}, 81^{-p}, \ldots, 81^{-p}, \ldots]$ and $C_2$ is the matrix

\[
C_2 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \cdots \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{bmatrix}.
\]

The matrix $E_pC_2E_p^{-1}$ can be better expressed in block lower triangular form as the matrix $C_p = [(\frac{2^j}{2^k})^{1-p}\frac{1}{2}M_{j,k}]$ where $M_{j,k}$ is the $2^j \times 2^k$ matrix each of whose entries is 1. That is

\[
C_p = \begin{bmatrix}
(\frac{1}{2})^{1-p}M_{0,0} & 0 & 0 & 0 & \cdots \\
(\frac{2}{2})^{1-p}\frac{1}{2}M_{1,0} & (\frac{2}{2})^{1-p}\frac{1}{2}M_{1,1} & 0 & 0 & \cdots \\
(\frac{4}{2})^{1-p}\frac{1}{2}M_{2,0} & (\frac{4}{2})^{1-p}\frac{1}{2}M_{2,1} & (\frac{4}{2})^{1-p}\frac{1}{2}M_{2,2} & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{bmatrix}.
\]

Let $\vec{v}_k$ be the unit vector $\vec{v}_k = \frac{1}{\sqrt{2^k}}(1, 1, \ldots, 1)^T$ and note that, for each $j$ and $k$, $M_{j,k} : \mathbb{C}^{2^k} \rightarrow \mathbb{C}^{2^j}$ is rank 1 with $\ker(M_{j,k})^\perp = \mathbb{C}\vec{v}_k$. Thus if $P_k : \mathbb{C}^{2^k} \rightarrow \mathbb{C}^{2^k}$ is the projection $P_k\vec{w}_k = \left\langle \vec{w}_k, \vec{v}_k \right\rangle \vec{v}_K$, then

\[
C_p \begin{pmatrix}
\vec{w}_0 \\
\vec{w}_1 \\
\vec{w}_2 \\
\vdots
\end{pmatrix} = C_p \begin{pmatrix}
P_0\vec{w}_0 \\
P_1\vec{w}_1 \\
P_2\vec{w}_2 \\
\vdots
\end{pmatrix}.
\]

Hence, for the purposes of determining the boundedness of $C_p$, it suffices to consider the action of $C_p$ on vectors of the form $\vec{x} = \begin{pmatrix}
\alpha_0\vec{v}_0 \\
\alpha_1\vec{v}_1 \\
\alpha_2\vec{v}_2 \\
\vdots
\end{pmatrix}$ where
\{\alpha_k\} \in l^2$. Since $M_{j,k} \vec{v}_k = 2^{\frac{k+j}{2}} \vec{v}_j$ we see that

\[
C_p \vec{x} = \begin{bmatrix}
(\frac{1}{4})^{1-p} \alpha_0 \vec{v}_0 \\
(\frac{3}{2})^{1-p} 2^{\frac{1}{2}} \frac{1}{2} \alpha_0 + (\frac{3}{2})^{1-p} \frac{1}{2} 2^{\frac{1}{2}} \frac{1}{2} \alpha_1 \vec{v}_1 \\
(\frac{3}{4})^{1-p} 2^{\frac{1}{2}} \frac{1}{2} \alpha_0 + (\frac{3}{2})^{1-p} \frac{1}{2} 2^{\frac{1}{2}} \frac{1}{2} \alpha_1 + (\frac{3}{4})^{1-p} \frac{1}{2} 2^{\frac{1}{2}} \frac{1}{2} \alpha_2 \vec{v}_2 \\
\vdots
\end{bmatrix}
\]

It is now apparent that $C_p$ is bounded if and only if the Toeplitz operator

\[
T_\phi = \begin{bmatrix}
1 & 0 & 0 & 0 \\
(2^{-p+\frac{1}{2}} - p) & 1 & 0 & 0 \\
2^{-2(p-\frac{1}{2})} & 2^{-(p-\frac{1}{2})} & 1 & 0 \\
2^{-3(p-\frac{1}{2})} & 2^{-2(p-\frac{1}{2})} & 2^{-(p-\frac{1}{2})} & 1
\end{bmatrix}
\]

with symbol $\phi(z) = \sum_{n=0}^{\infty} 2^{-n(p-\frac{1}{2})} z^n = \frac{1}{1 - 2^{-n(p-\frac{1}{2})} z}$ is bounded. Since $T_\phi$ is bounded if and only if $p > \frac{1}{2}$, the proof that $A_p = H^2(\mathbb{D})$ for $p > \frac{1}{2}$ is now complete. Additionally, we have shown that $A_p \neq H^2(\mathbb{D})$ if $p \leq \frac{1}{2}$. We next show that $A_p^\perp$ is $\{0\}$ if $p = \frac{1}{2}$ and $\mathbb{C}g_p$ if $p < \frac{1}{2}$. 
To this end, first note that
\[
f_{n+1}(1)f_n(z) - f_n(1)f_{n+1}(z) = (1 - b_{n+1})(1 - b_n)z^n - (1 - b_n)(1 - b_{n+1})z^{n+1}
\]
\[
= (1 - b_{n+1})z^n + (b_n b_{n+1} - 1)z^{n+1} + (1 - b_n)b_{n+1}z^{n+2}
\]
\[
= (1 - z)z^n((1 - b_{n+1}) - (1 - b_n)b_{n+1})
\]
\[
= (1 - z)g_n(z),
\]
where \(g_n(z) = z^n((1 - b_{n+1}) - (1 - b_n)b_{n+1})\) is in \(A_p \subset H^2(\mathbb{D})\). Suppose now that \(\phi_p = \sum_{n=0}^\infty \gamma_n z^n \in A_p^\perp\). Taking the \(H^2(\mathbb{D})\) inner product of \(\phi_p\) with \(g_n\) yields
\[
0 = \langle \phi_p, g_n \rangle = \gamma_n(1 - b_{n+1}) - \gamma_{n+1}(1 - b_n)b_{n+1}.
\]
Thus, for \(n = 0, 1, 2, \ldots\),
\[
\gamma_{n+1} = \frac{(1 - b_{n+1})}{(1 - b_n)} \frac{1}{b_{n+1}} \gamma_n
\]
which leads to
\[
\gamma_n = \frac{(1 - b_n)}{(1 - b_0)} \frac{1}{b_1 b_2 \cdots b_n} \gamma_0.
\]
Since \(b_1 b_2 \cdots b_n = (\frac{2}{n+2})^p\) and \(\frac{1 - b_0}{1 - b_n} \approx \frac{p}{(1 - b_0)n^2},\) we obtain
\[
\gamma_n \approx \frac{p}{(1 - b_0)} 2^{-p(n + 2)} (n + 2)^{p - 1} \gamma_0.
\]
It is now apparent that \(\{\gamma_n\} \in l^2\) if and only if \(p < \frac{1}{2}\). Since \(\gamma_n\) is comparable to \((1 - b_n)(n + 2)^p\), if we let \(g_p = \sum_{n=0}^\infty (1 - b_n)(n + 2)^p z^n\) for \(p < \frac{1}{2}\), then we have that
\[
A_p^\perp \subset \left\{ \begin{array}{ll} 0 & \text{if } p \geq \frac{1}{2} \\ \mathbb{C}g_p & \text{if } p < \frac{1}{2} \end{array} \right. .
\]
To complete the proof, it remains to show that \(A_p\) is the orthogonal complement of \(\{\mathbb{C}g_p\}\) if \(0 < p < \frac{1}{2}\). If \(g \in \{g_p\}^\perp\), then \(g(z) = \sum_{n=0}^\infty a_n z^n\) with
\[
\sum_{n=0}^\infty a_n(1 - b_n)(n + 2)^p = 0.
\]
We must show that \( f(z) = (1-z)g(z) = a_0 + \sum_{n=1}^{\infty} (a_n-a_{n-1})z^n \) is in \( H(K_p) \).

Note that if \( f \in H(K_p) \), then \( f(z) = \sum_{n=0}^{\infty} \beta_n f_n(z) = \sum_{n=0}^{\infty} \beta_n (1-b_n z)z^n = \beta_0 + \sum_{n=0}^{\infty} (\beta_n - \beta_{n-1} b_{n-1}) z^n \) for some sequence \( \{\beta_n\}_{n=0}^{\infty} \in l^2 \).

In order that this occur, we must have \( \beta_0 = a_0 \) and, for \( n \geq 1 \),

\[
\beta_n = a_n + \beta_{n-1} b_{n-1} - a_{n-1}.
\]

This recursion leads to \( \beta_1 = a_1 - (1-b_0)a_0 \) and, for \( n > 1 \),

\[
\beta_n = a_n - [(1-b_{n-1})a_{n-1} + b_{n-1}(1-b_{n-2})a_{n-2} + \cdots + (b_{n-1}b_{n-2}\cdots b_1)(1-b_0)a_0]
\]

\[
= a_n - \left[ (1-b_{n-1})a_{n-1} + \left( \frac{n}{n+1} \right)^p (1-b_{n-2})a_{n-2} + \left( \frac{n-1}{n+1} \right)^p (1-b_{n-3})a_{n-3} + \cdots + \left( \frac{2}{n+1} \right)^p (1-b_0)a_0 \right]
\]

\[
= a_n - \left( \frac{1}{n+1} \right)^p \left[ \sum_{k=0}^{n-1} a_k (1-b_k)(k+2)^p \right]
\]

\[
= a_n + \left( \frac{1}{n+1} \right)^p \left[ \sum_{k=n}^{\infty} a_k (1-b_k)(k+2)^p \right].
\]

This last equality follows from the fact that \( \sum_{n=0}^{\infty} a_n (1-b_n)(n+2)^p = 0 \).

We can express this in matrix form as

\[
\begin{pmatrix}
\beta_0 \\
\beta_1 \\
\beta_2 \\
\vdots
\end{pmatrix}
= (I + B_1)
\begin{pmatrix}
a_0 \\
a_1 \\
a_2 \\
\vdots
\end{pmatrix}
\]

where

\[
B_1 = \begin{bmatrix}
(1-b_0) \left( \frac{2}{3} \right)^p & (1-b_1) \left( \frac{3}{4} \right)^p & (1-b_2) \left( \frac{4}{5} \right)^p & (1-b_3) \left( \frac{5}{6} \right)^p & \cdots \\
0 & (1-b_1) \left( \frac{3}{2} \right)^p & (1-b_2) \left( \frac{4}{3} \right)^p & (1-b_3) \left( \frac{5}{4} \right)^p & \cdots \\
0 & 0 & (1-b_2) \left( \frac{4}{3} \right)^p & (1-b_3) \left( \frac{5}{4} \right)^p & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\]
As we observed earlier in the proof, \(1 - b_n \approx \frac{p}{n+2}\) for large \(n\). Hence \(B_1\) is bounded if and only if the matrix \(B_2\) is bounded where

\[
B_2 = \begin{bmatrix}
2^{p-1} & 3^{p-1} & 4^{p-1} & 5^{p-1} & \cdots \\
0 & 3^{p-1}2^{-p} & 4^{p-1}2^{-p} & 5^{p-1}2^{-p} & \cdots \\
0 & 0 & 4^{p-1}3^{-p} & 5^{p-1}3^{-p} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\]

Breaking \(B_2\) into blocks in the same manner as was done earlier, we see that the boundedness of \(B_2\) is equivalent to the boundedness of \(B_3\) where

\[
B_3 = \begin{bmatrix}
2^{p-1}M_{0,0} & 4^{p-1}M_{0,1} & 8^{p-1}M_{0,2} & 16^{p-1}M_{0,3} & \cdots \\
0 & 4^{p-1}2^{-p}M_{1,1} & 8^{p-1}2^{-p}M_{1,2} & 16^{p-1}2^{-p}M_{1,3} & \cdots \\
0 & 0 & 8^{p-1}4^{-p}M_{2,2} & 16^{p-1}4^{-p}M_{2,3} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\]

Recalling the estimate that \(||M_{n,m}|| = 2^{n+m}2^{-n-m}\) reduces the boundedness of \(B_3\) to the boundedness of the Toeplitz matrix

\[
T_\psi = \begin{bmatrix}
2^{p-\frac{3}{2}} & 2^{2p-\frac{3}{2}} & 2^{3p-\frac{3}{2}} & 2^{4p-\frac{3}{2}} & \cdots \\
0 & 2^{p-1} & 2^{2p-\frac{3}{2}} & 2^{3p-\frac{4}{2}} & \cdots \\
0 & 0 & 2^{p-1} & 2^{2p-\frac{3}{2}} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\]

Since the symbol \(\psi(z) = \sum_{n=0}^{\infty}2^n(p^{\frac{1}{2}})^{-\frac{1}{2}}z^n\) is bounded for \(p < \frac{1}{2}\), the Toeplitz matrix is bounded and the proof is complete. \(\square\)

The complete decomposition can now be summarized in the following corollary.

**Corollary 2.4.** The space \(H(K_p)\) decomposes as follows.

1. If \(p > \frac{1}{2}\), then \(H(K_p) = (1 - z)H^2(\mathbb{D}) + \mathbb{C}K_p(z, 1)\).
(2) If \( p = \frac{1}{2} \), then \( H(K_p) = (1 - z)A_p + CK_p(z, 1) \) where \( A_p \) is dense in \( H^2(\mathbb{D}) \), but not equal to \( H^2(\mathbb{D}) \).

(3) If \( 0 < p < \frac{1}{2} \), then \( H(K_p) = (1 - z)A_p + CK_p(z, 1) \) where \( A_p \) is the orthogonal complement in \( H^2(\mathbb{D}) \) of the function

\[
g_p(z) = \sum_{n=0}^{\infty} (1 - b_n)(n + 2)^p z^n.
\]

Recall that an analytic function \( \phi \) is a multiplier of \( H(K_p) \) if \( \phi f \in H(K_p) \) whenever \( f \in H(K_p) \). Our next goal is to give a characterization of the multipliers of \( H(K_p) \). Before doing so we establish a few simple facts about \( H(K_p) \).

**Proposition 2.5.** The following statements hold.

1. For \( p > 0 \), \( K_p(z, 1) \) extends continuously to \( \partial \mathbb{D} \).
2. If \( \frac{1}{2} < p < \infty \), then \( f(z) = 1 \) belongs to \( H(K_p) \).
3. If \( 0 < p < \infty \), then \( H(K_p) \subset H^2(\mathbb{D}) \).

**Proof.** Note that \( K_p(z, 1) = \sum_{n=0}^{\infty} f_n(1) f(z) \)

\[
= \sum_{n=0}^{\infty} \left( 1 - \left( \frac{n + 1}{n + 2} \right)^p \right) \left( 1 - \left( \frac{n + 1}{n + 2} \right)^p z \right) z^n
\]

\[
= 1 - \left( \frac{1}{2} \right)^p + \sum_{n=0}^{\infty} \left[ 1 - \left( \frac{n + 1}{n + 2} \right)^p - \left( \frac{n}{n + 1} \right)^p + \left( \frac{n}{n + 1} \right)^{2p} \right] z^n
\]

\[
= 1 - \left( \frac{1}{2} \right)^p + \sum_{n=0}^{\infty} \left[ 1 - \left( \frac{n}{n + 1} \right)^p \right]^2 + \left( \frac{n}{n + 1} \right)^p - \left( \frac{n + 1}{n + 2} \right)^p \right] z^n.
\]

Earlier it was observed that for large enough \( n \), \( 1 - \left( \frac{n}{n + 1} \right)^p < \frac{2p}{n+1} \). In similar fashion it is easy to verify that, for large \( n \),

\[
\left| \left( \frac{n}{n + 1} \right)^p - \left( \frac{n + 1}{n + 2} \right)^p \right| < \frac{2p}{(n + 1)(n + 2)}.
\]

Thus the series converges absolutely on \( \partial \mathbb{D} \) and part (1) of the proposition follows.
To establish (2), note that $1 = \sum_{n=0}^{\infty} \left( \frac{1}{n+1} \right)^p f_n(z)$ where
$$\{ f_n(z) = \left( 1 - \left( \frac{n+1}{n+2} \right)^p \right) z^n \}$$
is our set of orthonormal basis vectors. Since $\sum_{n=0}^{\infty} \left( \frac{1}{n+1} \right)^{2p} < \infty$ for $\frac{1}{2} < p < \infty$, $1 \in H(K_P)$.

Likewise it is easy to see that $H(K_p) \subset H^2(D)$ since
$$\sum_{n=0}^{\infty} \alpha_n f_n(z) = \alpha_0 + \sum_{n=1}^{\infty} \left[ \alpha_n - \left( \frac{n}{n+1} \right)^p \alpha_{n-1} \right] z^n$$
and the latter is in $H^2(D)$ whenever $\{ \alpha_n \}$ is an $l^2$ sequence. \hfill \Box

**Theorem 2.6.** For $p > \frac{1}{2}$, the function $\phi$ is a multiplier of $H(K_p)$ if and only if $\phi \in H^\infty$ and
$$\frac{\phi(z) - \phi(1)}{z-1} \in H^2(D).$$

**Proof.** Assume $\phi$ is a multiplier. Since $1 \in H(K_p)$, Corollary 2.4 allows us to write $\phi(z) - \phi(1) = (1 - z)g(z) + \alpha K_p(z,1)$ for some $g \in H^2$. Evaluating at $z = 1$ implies $\alpha = 0$. Therefore $\frac{\phi(z) - \phi(1)}{z-1} \in H^2(D)$.

In general Hilbert function spaces it is well known, see section 2.3, page 21 of [4], that if $\phi$ is a multiplier, then the multiplication operator $M_\phi$ is bounded, $K_p(z,\lambda)$ is an eigenvector of the adjoint $M_\phi^*$ with eigenvalue $\phi(\lambda)$, and consequently $\phi$ is bounded on the domain. Thus $\phi \in H^\infty$.

Conversely, assume that $\phi \in H^\infty$ and $\frac{\phi(z) - \phi(1)}{z-1} \in H^2(D)$. Clearly $\phi$ maps $(z-1)H^2(D)$ into $(z-1)H^2(D)$ and
$$\phi(z)K_p(z,1) = (z-1) \frac{\phi(z) - \phi(1)}{z-1} K_p(z,1) + \phi(1)K_p(z,1).$$

Since $K_p(z,1)$ is continuous on the closed disk $\overline{D}$, $\frac{\phi(z) - \phi(1)}{z-1} K_p(z,1)$ is in $H^2(D)$ and it follows from Corollary 2.4 that $\phi(z)$ multiplies $K_p(z,1)$ into $H(K_p)$.

\hfill \Box

**Corollary 2.7.** For $p > \frac{1}{2}$, the multiplication operator $M_z$ on $H(K_p)$ is bounded and similar to a rank one perturbation of the unilateral shift.
Proof. That $M_z$ is bounded is immediate from the theorem above. Part (1) of Corollary 2.4 establishes that $M_z$ is a rank one perturbation of the unilateral shift since $M_z$ acts as the unilateral shift on $(1 - z)H^2$.

\[ \square \]

**Theorem 2.8.** For $p > \frac{1}{2}$, $H(K_p)$ has the factorization property for $\lambda \in \mathbb{D}$: $f(\lambda) = 0$ implies $f(z) = (z - \lambda)g(z)$ for $g \in H(K_p)$.

**Proof.** Suppose $f(\lambda) = 0$ for some $\lambda \in \mathbb{D}$ and $f \in H(K_p)$. By Corollary 2.4

\[ f(z) = (1 - z)h(z) + f(1) \frac{K(z, 1)}{K(1, 1)} \]

for some $h \in H^2$. Hence $h(\lambda) = -\frac{f(1)}{1 - \lambda} \frac{K(\lambda, 1)}{K(1, 1)}$. Note

\[ g(z) = h(z) + \frac{f(1)}{1 - \lambda} \frac{K(z, 1)}{K(1, 1)} \in H^2 \]

and $g(\lambda) = 0$. Since $H^2$ has the factorization property, there exists $r \in H^2$ such that $g(z) = (z - \lambda)r(z)$. So

\[ h(z) = (z - \lambda)r(z) - \frac{f(1)}{1 - \lambda} \frac{K(z, 1)}{K(1, 1)} \]

and $(1 - z)h(z) = (z - \lambda)(1 - z)r(z) - (1 - z) \frac{f(1)}{1 - \lambda} \frac{K(z, 1)}{K(1, 1)}$.

\[ = f(z) - f(1) \frac{K(z, 1)}{K(1, 1)}. \]

Hence

\[ f(z) = (z - \lambda)(1 - z)r(z) + \left[-(1 - z) \frac{f(1)}{1 - \lambda} + f(1) \right] \frac{K(z, 1)}{K(1, 1)} \]

\[ = (z - \lambda)(1 - z)r(z) + f(1) \left[ \frac{(-1 + z + 1 - \lambda)}{(1 - \lambda)} \right] \frac{K(z, 1)}{K(1, 1)} \]

\[ = (z - \lambda) \left[(1 - z)r(z) + \frac{f(1)}{1 - \lambda} \frac{K(z, 1)}{K(1, 1)} \right]. \]

Since Corollary 2.4 implies $(1 - z)r(z) + \frac{f(1)}{1 - \lambda} \frac{K(z, 1)}{K(1, 1)}$ is in $H(K_p)$, factorization holds for all $\lambda \in \mathbb{D}$. \[ \square \]

The following corollary follows at once from Theorem 2.10 and Section 3 of Richter[11].
Corollary 2.9. If \( p > \frac{1}{2} \), then

1. \( M_z^* \) is in the Cowen-Douglas class \( B_1 \);
2. \( M_z \) is a cellular indecomposable operator;
3. The invariant subspaces of \( M_z \) are either of the form \( (1 - z)M \) where \( M = \psi H^2 \) for some inner function \( \psi \) or the span of the function \( K_p(z, 1) \) and a subspace of the form \( (1 - z)M \).

Theorem 2.10. If \( 0 < p < \frac{1}{2} \), then \( \phi \) is a non-trivial multiplier of \( H(K_p) \) if and only if

1. \( \phi \in H^\infty \);
2. \( \frac{\phi(z) - \phi(1)}{z - 1} K_p(z, 1) \) is in \( A_p \);
3. there exists a constant \( \lambda \in \mathbb{C} \) such that \( \langle \phi - \lambda, M_z^n g_p >_{H^2} 0 \) for all \( n \geq 0 \), where \( g_p(z) = \sum_{n=0}^{\infty} (1 - b_n)(n + 2)^p z^n \).

Proof. First, assume that \( \phi \) is a multiplier. As in the \( p > \frac{1}{2} \) case, it is well known (see [4]) that \( \phi \) is bounded. Since \( H(K_p) = (1 - z)A_p + \mathbb{C}K_p(z, 1) \) where \( A_p = H^2 \ominus \mathbb{C}g_p \), we can write

\[
\phi(z) \frac{K_p(z, 1)}{K_p(1, 1)} = (z - 1)h(z) + \phi(1) \frac{K_p(z, 1)}{K_p(1, 1)}.
\]

Hence

\[
\frac{\phi(z) - \phi(1)}{z - 1} K_p(z, 1) = K_p(1, 1)h(z) \in A_p.
\]

To establish the third condition, first note that if \( \phi \) is a multiplier, then it is easy to see that \( \phi A_p \subset A_p \). For simplicity we write the function \( g_p = \sum_{n=0}^{\infty} c_n z^n \) where \( c_n = (1 - b_n)(n + 2)^p \). Next, observe that for each \( n \geq 0 \), \( h_n(z) = -\frac{c_n}{c_0} + z^n \) is in \( A_p \) since \( h_n g_p >_{H^2} 0 \). Hence

\[
\phi(z)h_n(z) = -\frac{c_n}{c_0} \phi(z) + z^n \phi(z)
\]

is in \( A_p \) which implies \( 0 = \langle \phi(z)h_n(z), g_p(z) >_{H^2} \)

\[
= -\frac{c_n}{c_0} < \phi(z), g_p(z) >_{H^2} + < z^n \phi(z), g_p(z) >_{H^2}.
\]
Thus, for all \( n \geq 0 \),
\[
< z^n \phi(z), g_p(z) >_{H^2} = \frac{c_n}{c_0} < \phi(z), g_p(z) >_{H^2}.
\]

Note that \( \phi \) is a multiplier if and only if \( \phi - \lambda \) is also a multiplier for all \( \lambda \in \mathbb{C} \). Condition (3) results on letting \( \lambda \) be such that \( < \phi - \lambda, g_p >_{H^2} = 0 \).

For the converse, first note that since \( \phi \) is a multiplier if and only if \( \phi - \lambda \) is also a multiplier, we may reduce to the case where \( \lambda = 0 \). Next, note that conditions (1) and (3) imply that \( \phi(z)p(z) \) is orthogonal in \( H^2 \) to \( g_p \) for every polynomial \( p(z) \). Since the polynomials are dense in \( H^2 \), this means \( \phi(z)h(z) \) is in \( A_p \) for every \( h \in H^2 \). In particular, \( \phi A_p \subset A_p \). Since
\[
H(K_p) = (1 - z)A_p + \mathbb{C}K_p(z, 1),
\]
it suffices to show \( \phi(z)K_p(z, 1) \in H(K_p) \). By condition (2), \( \frac{\phi(z) - \phi(1)}{z-1} K_p(z, 1) \) is in \( A_p \). Hence
\[
[\phi(z) - \phi(1)]K_p(z, 1) = (z - 1)h(z)
\]
for some \( h \in A_p \). Thus \( \phi(z)K_p(z, 1) = (z - 1)h(z) + \phi(1)K_p(z, 1) \) is in \( H(K_p) \) and \( \phi \) is a multiplier.

\[ \square \]

**Corollary 2.11.** If \( 0 < p < \frac{1}{2} \), and \( g_p \) is a cyclic vector for \( M^*_z \), then \( H(K_p) \) has no non-trivial multipliers.

Although characterizations of the cyclic vectors for the backward shift exist in the literature (see Garcia [10]), applying the criteria to particular functions is often quite difficult. The authors were unable to determine whether or not \( g_p \) is a cyclic vector for \( M^*_z \) and must leave this as an open question.

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