

# Analytic Reproducing Kernels and Factorization

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**Abstract.** This paper relates questions about factorizations of positive matrices to properties of analytic reproducing kernel Hilbert spaces. In particular the question of when the polynomials are dense in a reproducing kernel Hilbert space is related to the factoring of an associated positive matrix into the form  $UU^*$  where  $U$  is an upper triangular matrix. An emphasis is placed on factorization and reproducing kernel methods as opposed to function theoretic methods.

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## Introduction

Let  $A = (a_{i,j})$  be the matrix of a positive operator on  $l^2$  with respect to the canonical basis. It is well known that such an operator may always be expressed as  $A = U^*U$  with  $U$  upper triangular. One way in which this can be done is by applying the Cholesky algorithm. What is less well known is that, in general, such an operator can not be expressed in the form  $A = UU^*$  with  $U$  upper triangular. This can be deduced from several places in the literature, most notably, S. C. Power [8], where the case of doubly infinite matrices is treated.

In this paper we relate questions about such factorizations to properties of the reproducing kernel Hilbert space  $H(A)$  with kernel  $K(z, w) = \sum_{i,j=0}^{\infty} a_{i,j} z^i \bar{w}^j$ .

In particular we prove that  $A$  factors as  $UU^*$  if and only if  $H(A)$  contains a dense set of polynomials. Moreover we show that every such  $A$  admits a unique decomposition  $A = A_1 + A_2$  where  $A_1 = U_1U_1^*$  and  $A_2$  is completely nonfactorable in the sense that if  $U_2U_2^* \leq A_2$  with  $U_2$  upper triangular, then  $U_2 = 0$ .

When  $A$  is the Toeplitz matrix of some positive  $L^\infty$  function on the circle, i.e.,  $A = T_\phi$ ,  $\phi \geq 0$ , then we show that  $U_1$  can be taken to be an analytic Toeplitz matrix  $U_1 = T_g$ . In this case we obtain a decomposition  $T_\phi = T_g^*T_g + T_\psi = T_{|g|^2} + T_\psi$ ,  $\psi \geq 0$ , and hence  $\phi = |g|^2 + \psi$ . This leads to another version of the Szego alternative: The reproducing kernel Hilbert space  $H(T_\phi)$  either contains a dense set of polynomials or no nonzero polynomials and this depends, respectively, on whether  $\log|\phi|$  is integrable or not.

Because of the connection with analytic reproducing kernel spaces our results extend readily to the multi-variable and operator-valued cases.

When  $A = (A_{i,j})$  is the matrix of a positive operator on  $l^2(n) \otimes \mathcal{C}$  for some fixed Hilbert space  $\mathcal{C}$ , then we obtain the analogous decomposition  $A = U_1U_1^* + A_2$  with  $U_1$  block upper triangular and  $A_2$  completely non-factorable. When we apply this to a positive operator valued function  $\Phi$  on the circle we show,  $\Phi = G^*G + \Psi$  where  $G$  extends to be analytic on the disk and  $\Psi$  is completely non-factorable. In this way, we obtain a new proof of the Wiener-Masani decomposition.

If  $A = (a_{I,J})$  is the matrix of an operator on  $l^2 \otimes \cdots \otimes l^2$  ( $n$  copies) with respect to the standard basis  $e_I = e_{i_1} \otimes \cdots \otimes e_{i_n}$ ,  $I = (i_1, \dots, i_n)$ , then as above  $A$  decomposes uniquely as  $A = A_1 + A_2$  where  $A_1 = U_1U_1^*$  with  $U_1$  “upper triangular” and  $A_2$  completely non-factorable. Here  $U_1 = (u_{I,J})$  “upper triangular” means that  $u_{I,J} = 0$  unless  $i_k \leq j_k$ , for all  $k = 1, \dots, n$ .

The results concerning the space  $H(T_\phi)$  are closely related to the recent work of Ben Lotto [4] and Don Sarason [10] concerning properties of de Brange spaces and their multipliers. There is also overlap with the work of de Brange, Rovnyak, and Rosenblum [3,9], as well as the extensive literature on factorization of positive operator valued functions going back to the work of Wiener and Masani [11]. For general properties of reproducing kernel Hilbert spaces the

reader is referred to [2,4,5].

Our paper seeks to emphasize the techniques of reproducing kernel Hilbert spaces and factorization of matrices as opposed to a function theoretic approach. The authors would like to thank Eric Nordgren for pointing out an error in a first draft preprint.

## Section 2. A Model for Analytic Reproducing Kernel Spaces

In [1] a range space model for analytic reproducing kernel spaces of functions defined on a neighborhood of 0 in the complex plane was given and studied. In this section we extend that model to include reproducing kernel spaces of functions that take on values in some Hilbert space and that are defined and analytic in some neighborhood of 0 in  $\mathbf{C}^n$ . Recall that if  $\mathcal{C}$  is a Hilbert space, and  $G$  is an open subset of  $\mathbf{C}^n$ , then by an *analytic reproducing kernel space of  $\mathcal{C}$ -valued functions on  $G$* , we mean a vector space of analytic  $\mathcal{C}$ -valued functions on  $G$  which is equipped with a norm making it a Hilbert space and which satisfies the additional property that for every  $w \in G$  the evaluation map  $E_w : H \rightarrow \mathcal{C}$  defined by  $E_w f = f(w)$  is a bounded linear map. In this case the map  $w \rightarrow E_w$  from  $G$  into  $\mathcal{B}(H, \mathcal{C})$  is analytic and the function  $K : G \times G \rightarrow \mathcal{B}(\mathcal{C})$  defined by  $K(z, w) = E_z E_w^*$  is analytic in  $z$  and coanalytic in  $w$ . The function  $K$  is called the *reproducing kernel* for  $H$ , and it is positive definite in the sense that, given  $z_1, \dots, z_n \in G$  and  $x_1, \dots, x_n \in \mathcal{C}$ ,

$$(*) \sum_{i,j=0}^n \langle K(z_i, z_j) x_j, x_i \rangle_{\mathcal{C}} \geq 0, \text{ where the inner product is taken in } \mathcal{C}.$$

Conversely given any  $K : G \times G \rightarrow \mathcal{B}(\mathcal{C})$  which is analytic in the first variable, coanalytic in the second, and which is positive definite (i.e., satisfies  $(*)$ ) then there is a unique analytic reproducing kernel space of  $\mathcal{C}$ -valued functions on  $G$  for which  $K$  is the reproducing kernel. This space is the completion of the functions of the form  $f(z) = \sum_{j=1}^n K(z, w_j) x_j$  for  $w_1, \dots, w_n \in G$ ,  $x_1, \dots, x_n \in \mathcal{C}$  arbitrary, with respect to the inner product  $\langle K(z, w_1) x_1, K(z, w_2) x_2 \rangle_H = \langle K(w_2, w_1) x_1, x_2 \rangle_{\mathcal{C}}$ . See [5] or [6] for details.

Let  $\mathbf{N} = \{0, 1, 2, \dots\}$  denote the set of non-negative integers and let  $n$  be a fixed positive integer, then the set  $\mathbf{N}^n$  is partially ordered by setting  $I = (i_1, \dots, i_n) \geq (j_1, \dots, j_n) = J$  if and only if  $i_k \geq j_k$  for  $k = 1, \dots, n$ . If  $z = (z_1, \dots, z_n) \in \mathbf{C}^n$  then we set  $z^I = z_1^{i_1} \dots z_n^{i_n}$ .

Let  $H$  be an analytic reproducing kernel space of  $\mathcal{C}$ -valued functions on  $G$  and assume that  $G$  contains 0. Then  $E_z$  has a power series expansion,  $E_z = \sum_{I \geq 0} z^I B_I$

, where  $B_I \in \mathcal{B}(H, \mathcal{C})$  and consequently,  $K(z, w) = \sum_{I, J \geq 0} z^I \bar{w}^J B_I B_J^*$  for  $z, w$  in

some neighborhood of 0. It is easy to see that the matrix  $A = (B_I B_J^*)_{I, J \geq 0}$  is formally positive, in the sense that if  $\{x_J\}_{J \in \mathbf{N}^n}$  is any collection of vectors in  $\mathcal{C}$ , with only finitely many non-zero terms, then  $\sum_{I, J} \langle B_I B_J^* x_J, x_I \rangle_{\mathcal{C}} \geq 0$ .

Conversely, if  $A = (A_{I,J})_{I,J \geq 0}$ ,  $A_{I,J}$  in  $\mathcal{B}(\mathcal{C})$ , is formally positive and  $K(z, w) = \sum z^I \bar{w}^J A_{I,J}$  converges on some polydisk, then  $K(z, w)$  is positive definite on that polydisk. Hence  $K_1(z, w) \leq K_2(z, w)$  if and only if  $(A_{I,J}^1) \leq (A_{I,J}^2)$ .

Let  $\{e_i\}_{i=0}^\infty$  denote the standard orthonormal basis for  $l^2$  and set  $l^2(n) = l^2 \otimes \cdots \otimes l^2$  ( $n$  copies) which has orthonormal basis  $e_I = e_{i_1} \otimes \cdots \otimes e_{i_n}$ ,  $I = (i_1, \dots, i_n) \in \mathbf{N}^n$ . Every vector  $x$  in  $l^2(n) \otimes \mathcal{C}$  has a unique representation as  $x = \sum_I e_I \otimes x_I$  with  $x_I \in \mathcal{C}$  and  $\sum \|x_I\|^2 < +\infty$ . If  $A \in \mathcal{B}(l^2(n) \otimes \mathcal{C})$  then  $A$  has a representation as  $A = (A_{I,J})$  where each  $A_{I,J} \in \mathcal{B}(\mathcal{C})$  and  $Ax = \sum_I e_I \otimes (\sum_J A_{I,J} x_J)$ . Moreover if  $A$  is a positive operator, then  $(A_{I,J})$  is formally positive, and  $K(z, w) = \sum z^I \bar{w}^J A_{I,J}$  converges for  $z$  and  $w$  in the unit polydisk  $\mathbf{D}^n$ .

Thus to every positive operator  $A = (A_{I,J})$  in  $\mathcal{B}(l^2(n) \otimes \mathcal{C})$  we have an associated analytic reproducing kernel Hilbert space of  $\mathcal{C}$ -valued functions on  $\mathbf{D}^n$  which we'll denote by  $H(A)$ .

Conversely, if  $H$  is an analytic reproducing kernel Hilbert space of  $\mathcal{C}$ -valued functions on a domain  $G$  in  $\mathbf{C}^n$ , then by a translation and rescaling we may assume that  $G$  contains the closed unit polydisk so that  $K(z, w) = \sum z^I \bar{w}^J A_{I,J}$  and one can show that  $A = (A_{I,J})$  defines a bounded positive operator on  $\mathcal{B}(l^2(n) \otimes \mathcal{C})$ .

Hence, letting  $A$  range over the bounded operators on  $\mathcal{B}(l^2(n) \otimes \mathcal{C})$ , up to some equivalence, one obtains all the reproducing kernel Hilbert spaces of  $\mathcal{C}$ -valued functions on any domain  $G$  in  $\mathbf{C}^n$ . What we now wish to do is describe a model for these as range spaces which extends the model described in [1].

Recall that if  $\mathcal{M}$  is a Hilbert space and  $B \in \mathcal{B}(\mathcal{M})$ , then the *range space*  $\mathcal{R}(B)$  is the Hilbert space one obtains by equipping the range of  $B$  with the norm,  $\|y\|_{\mathcal{R}(B)} = \|x\|_{\mathcal{M}}$ , where  $x$  is the unique vector in  $\ker(B)^\perp$  satisfying  $y = Bx$ . Thus,  $\mathcal{R}(B)$  is a vector subspace of  $\mathcal{M}$ , but with a different norm.

**Theorem 2.1.** Let  $A = (A_{I,J})$  be a bounded positive operator in  $\mathcal{B}(l^2(n) \otimes \mathcal{C})$ , so that  $H(A)$  is an analytic reproducing kernel Hilbert space of  $\mathcal{C}$ -valued functions on  $\mathbf{D}^n$ . If  $A = BB^*$  with  $B \in \mathcal{B}(l^2(n) \otimes \mathcal{C})$ , then the map  $U : \mathcal{R}(B) \rightarrow H(A)$  defined by  $U(\sum e_I \otimes y_I) = \sum_I z^I y_I$  is unitary.

**Proof.** Consider the space of power series  $H = \{f(z) = \sum_I z^I y_I : y \in \mathcal{R}(B)\}$  endowed with the norm,  $\|f\| = \|y\|_{\mathcal{R}(B)}$ . If  $y = Bx$  and  $w = (w_1, \dots, w_n) \in \mathbf{D}^n$ , then  $\|\sum w^I y_I\|_{\mathcal{C}}^2 \leq \|\{w^I\}\|_{l^2(n)} \|\{\|y_I\|\}\|_{l^2(n)} \leq (\sum |w|^{2I})^{1/2} (\sum \|y_I\|^2)^{1/2} \leq (\prod_{i=1}^n \frac{1}{1 - |w_i|^2})^{1/2} \cdot \|B\| \cdot \|x\| = (\prod_{i=1}^n \frac{1}{1 - |w_i|^2})^{1/2} \cdot \|B\| \cdot \|y\|_{\mathcal{R}(B)}$ . Hence  $H$  is a space of  $\mathcal{C}$ -valued analytic functions on  $\mathbf{D}^n$  and evaluation at  $w$  is a bounded linear functional for each  $w$  in  $\mathbf{D}^n$ .

By the definition of  $H$ , the map  $U : \mathcal{R}(B) \rightarrow H$  defined as above is unitary and so we can complete the proof by showing that  $H = H(A)$ . To do this it will be sufficient to show that the two spaces have the same reproducing kernel.

To this end let  $v \in \mathcal{C}$  and let  $f(z) = \sum_I z^I f_I$  be in  $H$  where  $f_I = \sum_J B_{I,J} y_J$ . Recall  $E_z : H \rightarrow \mathcal{C}$  is given by  $E_z h = \sum_I z^I h_I$  for each  $z \in \mathbf{D}^n$  and  $h \in H$ . Thus if  $w \in \mathbf{D}^n$ , then  $E_w^* : \mathcal{C} \rightarrow H$  and there exists  $x$  in  $l^2(n) \otimes \mathcal{C}$  such that  $Bx = E_w^* v$ . Hence  $\langle x, y \rangle_{l^2(n) \otimes \mathcal{C}} = \langle E_w^* v, f \rangle_H = \langle v, \sum_I w^I f_I \rangle_{\mathcal{C}} = \langle v, \sum_{I,J} w^I B_{I,J} y_J \rangle_{\mathcal{C}} = \sum_{I,J} \langle \bar{w}^I v, B_{I,J} y_J \rangle_{\mathcal{C}} = \sum_J \langle \sum_I B_{I,J}^* \bar{w}^I v, y_J \rangle_{\mathcal{C}} = \langle B^* \hat{v}, y \rangle_{l^2(n) \otimes \mathcal{C}}$  where  $\hat{v} = \sum_I e_I \otimes \bar{w}^I v \in l^2(n) \otimes \mathcal{C}$ . Since  $y \in (\ker(B))^\perp$ ,  $\langle B^* \hat{v}, y \rangle_{l^2(n) \otimes \mathcal{C}} = \langle PB^* \hat{v}, y \rangle_{l^2(n) \otimes \mathcal{C}}$  where  $P : l^2(n) \otimes \mathcal{C} \rightarrow (\ker(B))^\perp$ . Hence  $x = PB^* \hat{v}$  and  $Bx = BPB^* \hat{v} = A\hat{v} = \sum_{I,J} e_I \otimes A_{I,J} \bar{w}^J v$ . Thus  $E_z E_w^* v = \sum_{I,J} z^I A_{I,J} \bar{w}^J v$ . This shows the two kernels are the same.  $\square$

**Remark.** The above theorem gives a very concrete condition for a power series to belong to the space  $H(A)$ . Namely  $\sum z^I v_I \in H(A)$  if and only if  $\sum e_I \otimes v_I \in \mathcal{R}(A^{1/2})$ . It is useful to recall that if  $BB^* = CC^*$  then  $\mathcal{R}(B) = \mathcal{R}(C)$ .

Let  $S_k \in \mathcal{B}(l^2(n) \otimes \mathcal{C})$  denote the shift operator in the  $k$ -th component, i.e.,  $S_k(e_I \otimes v) = e_{I+\delta_k} \otimes v$  where  $\delta_k \in \mathbf{N}^n$  is the vector which is 1 in the  $k$ -th component and 0 elsewhere.

**Theorem 2.2.** If  $A \in \mathcal{B}(l^2(n) \otimes \mathcal{C})$  is a bounded positive operator, with  $A = BB^*$ , then multiplication by the  $k$ -th coordinate function  $z_k$  is a bounded operator on  $H(A)$  if and only if  $S_k \mathcal{R}(B) \subset \mathcal{R}(B)$ . Moreover if this is the case, and  $U$  denotes the unitary operator of Theorem 2.1, then  $M_{z_k} U = U S_k$ .

**Proof.** If  $S_k \mathcal{R}(B) \subset \mathcal{R}(B)$ , then by the Closed Graph Theorem  $S_k$  defines a bounded positive operator on  $\mathcal{R}(B)$ . Hence  $U S_k U^{-1}$  is a bounded operator on  $H(A)$  which is easily seen to be  $M_{z_k}$ . The remaining facts follow similarly.  $\square$

### Section 3. Reproducing Kernels and Factorization

Let  $U = (U_{I,J})$ ,  $I, J \in \mathbf{N}^n$ ,  $U_{I,J} \in \mathcal{B}(\mathcal{C})$  be a bounded operator on  $l^2(n) \otimes \mathcal{C}$ . We shall call  $U$  *n-upper triangular* if  $U_{I,J} = 0$  unless  $I \leq J$ . When  $n = 1$  and  $\dim(\mathcal{C}) = 1$  this definition reduces to the usual definition of upper triangular and is equivalent to requiring that  $U \in \text{alg}(\mathcal{N})$  where  $\mathcal{N}$  is the nest of projections  $\mathcal{N} = \{P_0, P_1, P_2, \dots\}$  with  $P_k$  denoting the projection onto the span of  $\{e_0, e_1, \dots, e_k\}$ . If  $\dim(\mathcal{C}) = 1$  and  $n > 1$ , then  $U$  is *n-upper triangular* on  $l^2(n)$  if and only if  $U \in \text{alg}(\mathcal{N}) \otimes \dots \otimes \text{alg}(\mathcal{N})$  ( $n$  copies). Thus, when  $\dim(\mathcal{C}) > 1$

our definition of  $U$  being  $n$ -upper triangular is equivalent to requiring that  $U$  belong to a tensor of nest algebras with the nest of multiplicity  $\dim(\mathcal{C})$ .

We call a function  $f : \mathbf{C}^n \rightarrow \mathcal{C}$  a *polynomial* if there exists a *finite* collection of vectors  $v_i \in \mathcal{C}$ ,  $I \in \mathbf{N}^n$  such that  $f(z) = \sum z^I v_i$ . We call  $\max\{|I| : v_i \neq 0\}$  the *degree* of  $f$ , where  $|I| = i_1 + \dots + i_n$ .

**Theorem 3.1 .** Let  $A = (A_{I,J})$  be a bounded positive operator on  $l^2(n) \otimes \mathcal{C}$  and let  $H(A)$  denote the analytic reproducing kernel Hilbert space of  $\mathcal{C}$ -valued functions with kernel  $K(z, w) = \sum z^I \bar{w}^J A_{I,J}$ . Then there exists a bounded operator on  $l^2(n) \otimes \mathcal{C}$ ,  $U = (U_{I,J})$  which is  $n$ -upper triangular such that  $A = UU^*$  if and only if the polynomials in  $H(A)$  are dense in  $H(A)$ .

**Proof.** Assume that  $A = UU^*$  with  $U$   $n$ -upper triangular. Let  $P_k$  be the projection of  $(l^2(n) \otimes \mathcal{C})$  onto the span of  $\{e_I \otimes v : |I| \leq k, v \in \mathcal{C}\}$ . The isomorphism of Theorem 2.1 carries  $y \in \mathcal{R}(U)$  to a polynomial of degree less than or equal to  $k$  exactly when  $P_k y = y$ . Thus to prove that the polynomials are dense in  $H(A)$  is equivalent to proving that for every  $y \in \mathcal{R}(U)$  there exists  $y_k \in \mathcal{R}(U) \cap \mathcal{R}(P_k)$  such that  $\|y - y_k\|_{\mathcal{R}(U)} \rightarrow 0$ . Let  $y = Ux$  and set  $x_k = P_k x$ . Since  $U$  is  $n$ -upper triangular  $P_k U x_k = U x_k$ . Thus  $y_k = U x_k \in \mathcal{R}(U) \cap \mathcal{R}(P_k)$  and  $\|y - y_k\|_{\mathcal{R}(U)} \leq \|x - x_k\| \rightarrow 0$  as  $k \rightarrow \infty$ .

Conversely, assume that the polynomials are dense in  $H(A)$ . Let  $H_k \subseteq H(A)$  be the subspace spanned by polynomials of degree at most  $k$ . Then  $H_k$  is an analytic reproducing kernel Hilbert space and has a reproducing kernel  $K_k(z, w)$  that can be expanded as a power series  $K_k(z, w) = \sum z^I \bar{w}^J A_{I,J}^k$ . It is not hard to see that  $A_{I,J}^k = 0$  unless  $|I| \leq k$  and  $|J| \leq k$ , and that  $A_k = (A_{I,J}^k)$  is a bounded positive operator on  $l^2(n) \otimes \mathcal{C}$ , with  $A_k \leq A_{k+1} \leq A$ .

Since  $\bigcup_{k \geq 0} H_k$  is dense in  $H(A)$  we have that  $A$  is the strong limit of the  $A_k$ 's.

To see this, first note that if  $Ay - A_k y = A^{1/2} x$  then  $\|Ay - A_k y\| \leq \|A^{1/2}\| \|x\|$  and so  $\|Ay - A_k y\| \leq \|A^{1/2}\| \|Ay - A_k y\|_{\mathcal{R}(A^{1/2})}$ . Since  $A = A_k + B_k$ ,  $B_k = A - A_k$  corresponds to an orthogonal decomposition of  $H(A)$  we have that  $H(A_k) \cap H(B_k) = (0)$ , i.e., that  $\mathcal{R}(A_k^{1/2}) \cap \mathcal{R}(B_k^{1/2}) = (0)$ . Finally since  $Ay = A^{1/2}(A^{1/2}y) \in \mathcal{R}(A^{1/2})$  decomposes as  $Ay = A_k y + B_k y$  we see that the orthogonal projection of the function in  $H(A)$  corresponding to  $Ay$  onto  $H(A_k)$  is just  $A_k y$ . But since the polynomials in  $H(A)$  are dense,  $\|Ay - A_k y\|_{\mathcal{R}(A^{1/2})} \rightarrow 0$ .

Set  $C_0 = A_0$  and for  $k \geq 1$ ,  $C_k = A_k - A_{k-1}$  so that each  $C_k \geq 0$  and  $A = \sum_k C_k$  in the strong topology. If we let  $C_k = (C_{I,J}^k)$ , then  $C_{I,J}^k = 0$  unless  $|I| \leq k$  and  $|J| \leq k$ .

Factor  $C_k = B_k B_k^*$  with  $B_k = (B_{I,J}^k)$  and  $B_{I,J}^k = 0$  unless  $|I| \leq k$  and  $|J| \leq k$  (so for example we could set  $B_k = C_k^{1/2}$ ).

Let  $\delta = (1, \dots, 1) \in \mathbf{N}^n$  and define a sequence of integers  $\{m_k\}$  by  $m_0 = 0$ ,  $m_{k+1} = m_k + k + 1$ . Define  $U_k = (U_{I,J}^k)$  by setting  $U_0 = B_0$ ,  $U_{I,J}^k = 0$  for  $J < m_k \delta$ , and  $U_{I,J}^k = B_{I, J - m_k \delta}^k$  for  $J \geq m_k \delta$ . That is  $U_k$  is the matrix obtained from  $B_k$  by translating the  $J$ -th column of  $B_k$ ,  $m_k \cdot \delta$  units. Since  $B_{I,J}^k = 0$

unless  $|I| \leq k$  (which implies  $I \leq k\delta$ ) and  $U_{I,J}^k = 0$  unless  $J \geq m_k \cdot \delta$ , we have that  $U_{I,J}^k = 0$  unless  $I \leq J$  and hence  $U_k$  is  $n$ -upper triangular for all  $k$ .

Note that  $U_{I,J}^k = 0$  unless  $m_k\delta \leq J \leq (m_k + k)\delta$  and that for  $k \neq j$  no two of these order intervals intersect. Hence,  $U_k U_j^* = 0$  for  $k \neq j$ .

Thus, if we let  $U = \sum_{k=0}^{\infty} U_k$  be the strong sum of these operators, then  $U$  is  $n$ -upper triangular and  $UU^* = \sum_{k=0}^{\infty} U_k U_k^* = \sum_{k=0}^{\infty} B_k B_k^* = A$ , which completes the proof of the theorem.  $\square$

**Theorem 3.2.** Let  $A$  be a bounded positive operator on  $l^2(n) \otimes \mathcal{C}$ . Then there exists a unique positive operator  $A_1$  satisfying:

- i)  $A_1 \leq A$ ;
- ii)  $A_1$  factors as  $UU^*$  with  $U$   $n$ -upper triangular;
- iii) if  $W$  is  $n$ -upper triangular and  $WW^* \leq (A - A_1)$ , then  $W = 0$ ;
- iv) if  $U$  is  $n$ -upper triangular and  $UU^* \leq A$ , then  $UU^* \leq A_1$ .

**Proof.** Let  $H_1 \subseteq H(A)$  denote the closure of the polynomials in  $H(A)$  and set  $H_1 = H(A_1)$  so that  $A_1 \leq A$ . By Theorem 3.1,  $A_1$  factors as  $UU^*$  with  $U$   $n$ -upper triangular.

Let  $A_2 = A - A_1$ . Since  $H(A_1) \subseteq H(A)$  and the norms agree, we have that  $H(A_2) = H(A) \ominus H(A_1)$ . Now if  $WW^* \leq A_2$ , then  $H(WW^*) \subseteq H(A_2) \subseteq H(A)$ . But by Theorem 3.1, the polynomials are dense in  $H(WW^*)$  and hence  $H(WW^*) \subseteq H(A_1)$ , from which we obtain that  $H(WW^*) = 0$  and so  $WW^* = 0$ .

If  $UU^* \leq A$ , then  $H(UU^*) \subseteq H(A)$  and since  $U$  is  $n$ -upper triangular, the polynomials in  $H(UU^*)$  are dense in  $H(UU^*)$  in the  $H(UU^*)$ -norm. But the  $H(UU^*)$ -norm dominates the  $H(A)$ -norm and hence  $H(UU^*)$  is contained in the closure of the polynomials in  $H(A)$ , which is  $H(A_1)$ . Since  $H(UU^*) \subseteq H(A_1)$  and for  $f \in H(UU^*)$ ,  $\|f\|_{H(UU^*)} \geq \|f\|_{H(A)} = \|f\|_{H(A_1)}$ , we have that  $UU^* \leq A_1$ .

Finally, assume that  $A_2$  has properties i) through iv). Then  $A_2 = U_2 U_2^*$  and so by iv),  $A_2 \leq A_1$ . Similarly,  $A_1 \leq A_2$  and so  $A_1 = A_2$ .  $\square$

We now wish to relate our decomposition of a positive operator to the Wold-Zasuhin decomposition of Power[8]. To this end let  $C = (C_{i,j})_{i,j=-\infty}^{+\infty}$  be a positive operator on  $\sum_{j=-\infty}^{+\infty} \oplus \mathcal{C}_j$  where each  $\mathcal{C}_j = \mathcal{C}$ . Power[8] provides a decomposition

of  $C = C_{-\infty} + U^*U$  where  $U = (U_{i,j})_{i,j=-\infty}^{+\infty}$  is upper triangular, i.e.,  $U_{i,j} = 0$  unless  $i \leq j$ , and  $C_{-\infty}$  is positive.

To see the relationship between this decomposition and ours, let  $A = (A_{i,j})_{i,j=0}^{+\infty}$  be a positive operator on  $l^2 \otimes \mathcal{C} = \sum_{j=-\infty}^{+\infty} \oplus \mathcal{C}_j$ ,  $\mathcal{C}_j = \mathcal{C}$  and define  $C = (C_{i,j})_{i,j=-\infty}^{+\infty}$

by  $C_{i,j} = 0$  if  $i > 0$  or  $j > 0$  and  $C_{i,j} = A_{-i,-j}$  for  $i \leq 0$  and  $j \leq 0$ . Then  $C$  is a positive operator and by Power[8],  $C = C_{-\infty} + U^*U$ ,  $C_{-\infty} = (B_{i,j})$ ,  $U = (U_{i,j})$ . Since  $C_{i,i} = 0$  for  $i > 0$ ,  $u_{i,j} = B_{i,j} = 0$  unless  $i \leq 0$  and  $j \leq 0$ . Substituting  $-i$  for  $i$  and  $-j$  for  $j$  makes  $U$  lower triangular and we obtain,  $A = (B_{-i,-j})_{i,j=0}^{+\infty} + L^*L$ , where  $L = (U_{-i,-j})_{i,j=0}^{+\infty}$  and  $L^*$  is upper triangular. Thus the Wold-Zasuhin decomposition yields a decomposition of  $A = R + VV^*$  with  $V$  upper triangular and  $R$  positive. We shall prove that  $VV^* = A_1$  of Theorem 3.2.

We start by recalling Power's construction. If  $\mathcal{H}$  is a Hilbert space with an orthogonal decomposition  $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$  and  $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$  is the matrix of a positive operator relative to this decomposition, then the strong limit of  $b(c + \frac{1}{n})^{-1/2}$  exists and defines an operator  $d$ , satisfying  $dc^{1/2} = b$ . The operator  $R = \Theta \begin{pmatrix} 0 & d \\ 0 & c^{1/2} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ d^* & c^{1/2} \end{pmatrix} = \begin{pmatrix} dd^* & b \\ b^* & c \end{pmatrix}$  is less than  $A$  and among all positive operators of the form  $B = \begin{pmatrix} x & b \\ b^* & c \end{pmatrix}$  satisfying  $B \leq A$  it is minimal [8, Lemma 1]. Power calls  $R$  the  $\mathcal{H}_2$ -minimal part of  $A$ . Consequently, among all positive operators of the form  $P = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$  which are less than  $A$  we see that the maximal one is given by taking  $p = a - dd^*$ .

Assume we are given a Hilbert space  $\mathcal{C}$  and a positive operator  $A = (A_{I,J})$  on  $l^2(n) \otimes \mathcal{C}$ . If  $k$  is a non-negative integer, let  $\mathcal{C}(k) = \sum_{|I| \leq k} \oplus \mathcal{C}_I$  and relative to the decomposition  $l^2(n) \otimes \mathcal{C} = \mathcal{C}(k) \oplus \mathcal{C}(k)^\perp$ , let  $R_k = (R_{I,J}(k))$  denote the  $\mathcal{C}(k)^\perp$ -minimal part of  $A$  and set  $P_k = (P_{I,J}(k)) = A - R_k \geq 0$ . Then  $P_{I,J}(k) = 0$  unless  $|I| \leq k$  and  $|J| \leq k$ . Since the compression of  $R_k$  to  $\mathcal{C}(k+1)^\perp$  is equal to the compression of  $A$  to  $\mathcal{C}(k+1)^\perp$ , by the minimality condition  $R_k \geq R_{k+1}$ . We let  $R$  denote the strong limit of this decreasing sequence of positive operators. When  $n = 1$  this is the construction employed by Power in his construction of the Wold-Zasuhin operator. We wish to connect this operator with our reproducing kernel Hilbert space constructions.

**Lemma 3.3** Let  $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$  be the matrix of a positive operator on  $\mathcal{H}_1 \oplus \mathcal{H}_2$  and let  $R = \begin{pmatrix} 0 & d \\ 0 & c^{1/2} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ d^* & c^{1/2} \end{pmatrix}$  be the  $\mathcal{H}_2$ -minimal part of  $A$ . Then  $\mathcal{R}(R^{1/2}) \cap \mathcal{H}_1 = (0)$  and  $\mathcal{R}((A - R)^{1/2}) \subseteq \mathcal{H}_1$ .

**Proof.** Since  $A - R = \begin{pmatrix} a - dd^* & 0 \\ d^* & 0 \end{pmatrix}$ ,  $\mathcal{R}((A - R)^{1/2}) \subseteq \mathcal{H}_1$ . Since  $\mathcal{R}(R^{1/2}) = \mathcal{R}\left(\begin{pmatrix} 0 & d \\ 0 & c^{1/2} \end{pmatrix}\right)$ , it will be enough to prove that if  $c^{1/2}h_2 = 0$ , then  $dh_2 = 0$ . But since  $\begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$  is positive,  $ch_2 = 0$  implies  $bh_2 = 0$ . Thus, if  $c^{1/2}h_2 = 0$ ,

then  $b(c + \frac{1}{n})^{-1/2}h_2 = 0$  and hence  $dh_2 = 0$ .  $\square$

**Theorem 3.4.** Let  $A = (A_{I,J})$  be a positive operator on  $l^2(n) \otimes \mathcal{C}$ , let  $R_k$  denote the  $\mathcal{C}(k)^\perp$ - minimal part of  $A$ , let  $H_k$  denote the subspace of  $H(A)$  consisting of polynomials of degree at most  $k$  and let  $P_k$  be the matrix of its reproducing kernel. Then  $A = P_k + R_k$ , and consequently  $H(R_k) = H_k^\perp$ .

**Proof.** We must prove that  $H(A - R_k) = H_k$  and that the norms agree.

We have that  $H(A) = H(A - R_k) + H(R_k)$ . By Lemma 3.3,  $\mathcal{R}((A - R_k)^{1/2}) \cap \mathcal{R}(R_k^{1/2}) = (0)$  and hence by Aronszajn[2, page 353],  $H(A - R_k) \perp H(R_k)$ , and since  $\mathcal{R}((A - R_k)^{1/2}) \subseteq \mathcal{C}(k)$  we have that  $H(A - R_k) \subseteq H_k$ , isometrically. If  $H(A - R_k) \neq H_k$  then there would exist  $f \in H_k \cap H(A - R_k)^\perp$  so that  $f \in H_k \cap H(R_k)$ . If we write  $f = \sum_{|I| \leq k} z^I v_I$ , then by Theorem 2.1,  $\sum_{|I| \leq k} e_I \otimes v_I \in \mathcal{C}(k) \cap \mathcal{R}(R_k^{1/2})$ . However, again by Lemma 3.3  $\mathcal{C}(k) \cap \mathcal{R}(R_k^{1/2}) = (0)$ . Consequently  $H(A - R_k) = H_k$  isometrically, and so  $A - R_k = P_k$ , which completes the proof of the theorem.  $\square$

**Corollary 3.5.** If  $A = (A_{I,J})$  is a positive operator on  $l^2(n) \otimes \mathcal{C}$ ,  $R$  denotes the Wold-Zasuhin operator for  $A$ , and  $P$  denotes the operator for the closure of the space of polynomials in  $H(A)$ , then  $A = R + P$ .

**Proof.** Since  $P_k$  increases strongly to  $P$ , we have that  $A - P_k = R_k$  decreases strongly to both  $R$  and to  $A - P$ , hence  $A - P = R$ .  $\square$

**Proposition 3.6.** Let  $A = (A_{I,J})$  be a positive operator on  $l^2(n) \otimes \mathcal{C}$ . The following are equivalent:

- i)  $H(A)$  contains no polynomials;
- ii) if  $U$  is  $n$ -upper triangular and  $UU^* \leq A$ , then  $U = 0$ ;
- iii) for every  $k$ , the  $\mathcal{C}(k)^\perp$ -minimal part of  $A$  is  $A$ ;
- iv)  $A = R$ , the Wold-Zasuhin part of  $A$ .

**Proof.** If  $UU^* \leq A$ , then  $H(UU^*) \subseteq H(A)$ . But since  $H(U)$  clearly contains the polynomials, we see the equivalence of i) and ii).

By Theorem 3.2,  $H(A)$  contains no polynomials if and only if  $A_1 = 0$  and hence if and only if  $A = R$ , the Wold-Zasuhin operator for  $A$ . But for each  $k$ ,  $A \geq R_k \geq R$  where  $R_k$  is the  $\mathcal{C}(k)^\perp$ - minimal part of  $A$ . Hence,  $H(A)$  contains no polynomials if and only if  $A = R_k = R$  for all  $k$ . The equivalence of iii) and iv) is clear.  $\square$

Using the reproducing kernel methods it is fairly easy to produce examples of the above decomposition.

**Example 3.7a)** Let  $a_0, a_1, \dots$  be a square summable sequence, and let  $A = (a_i \bar{a}_j)$ . Then  $H(A)$  is the 1-dimensional subspace of  $H^2(\mathbf{D})$  spanned by  $\sum a_i z^i$ . Hence if infinitely many of the  $a_i$ 's are nonzero then  $A = R$  and we are in the case of Proposition 3.6.

**b)** Let  $g(z) = a_0 + a_1 z + \dots$  be analytic on  $\mathbf{D}$  and chosen such that  $g \cdot H^2(\mathbf{D})$  contains no polynomials. For example  $g$  could be any nonzero analytic function which vanishes on a non- Blaschke sequence. Then setting  $\|gf\| = \|f\|_{H^2(\mathbf{D})}$  makes  $g \cdot H^2(\mathbf{D})$  a reproducing kernel space of analytic functions on  $\mathbf{D}$  with kernel  $K(z, w) = \frac{g(z)\overline{g(w)}}{1-z\bar{w}}$  [1, Proposition 1.6]. If for  $r < 1$  we let  $H = \{g(rz)f(rz) : f \in H^2\}$  then  $H$  is a reproducing kernel space of functions on the disk of radius  $r^{-1}$ , with kernel  $\frac{g(rz)\overline{g(rw)}}{1-r^2 z\bar{w}}$  and  $H = H(A)$  where  $A = D_r T_{g_r} T_{g_r}^* D_r$  is a bounded positive operator on  $l^2$ . Here  $g_r(e^{i\theta}) = g(re^{i\theta})$ , and  $D_r$  is the diagonal operator whose  $k$ -th diagonal entry is  $r^k$ . Since  $H(A)$  contains no polynomials, we have  $A = R$ .

**c)** Let  $g$  be as above and consider  $H = gH^2 \oplus H^2$  with  $\|(gf_1, f_2)\|^2 = \|f_1\|^2 + \|f_2\|^2$ . Then [1, Example 1.6]  $H$  is a reproducing kernel space on  $\mathbf{D}$  with kernel  $\frac{1+g(z)\overline{g(w)}}{1-z\bar{w}}$ . If for  $r < 1$  we let  $H_r = \{(g(rz)f_1(rz), f_2(rz))\}$  as above, then  $H_r = H(B)$  where  $B = D + A$ , with  $D$  the diagonal operator with diagonal entries  $d_{i,i} = r^{2i}$  and  $A$  as in b). The operator  $B$  is a bounded positive operator on  $l^2$  and we have that  $B = A_1 + R$  with  $A_1 = D$ ,  $R = A$ .  $\square$

We close this section with a couple of results on the Wold-Zasuhin part of an operator.

**Proposition 3.8.** Let  $A = (A_{i,j})$  be a positive operator on  $l^2 \otimes \mathcal{C}$  of finite band width. Then the Wold-Zasuhin part of  $A$  is 0 and consequently the polynomials are dense in  $H(A)$ .

**Proof.** Suppose  $A_{i,j} = 0$  for  $|i - j| \geq n$ . If we apply the usual Cholesky algorithm we obtain a factorization  $A = LL^*$  with  $L = (L_{i,j})$  lower triangular, i.e,  $L_{i,j} = 0$  if  $i < j$ . Moreover  $L$  has the same finite band width,  $L_{i,j} = 0$  if  $i \geq j + n$ . Define an operator  $U = (U_{i,j})$  by setting  $U_{i,j} = 0$  for  $j < n$  and  $U_{i,j} = L_{i,j-n}$  for  $j \geq n$ , i.e., the  $j$ -th column of  $L$  is the  $(j + n)$ -th column of  $U$ . Then  $U$  is upper triangular and  $UU^* = A$ , hence the Wold-Zasuhin part of  $A$  must be 0.  $\square$

**Proposition 3.9.** Let  $A = (A_{i,j})$  be a positive operator on  $l^2 \otimes \mathcal{C}$  and let  $S = S_1 \otimes 1_{\mathcal{C}}$  where  $S_1$  is the unilateral shift on  $l^2$ . If  $R$  is the Wold-Zasuhin part of  $A$ , then  $S^*RS$  is the Wold-Zasuhin part of  $S^*AS$ . Consequently, if the polynomials are dense in  $H(A)$ , then they are dense in  $H(S^*AS)$ .

**Proof.** Recall the construction of  $R$ , we obtain it as the limit of  $\begin{pmatrix} V_k V_k^* & B_k \\ B_k^* & C_k \end{pmatrix}$  where  $\begin{pmatrix} A_k & B_k \\ B_k^* & C_k \end{pmatrix}$  is the decomposition of  $A$  relative to  $\mathcal{C}(k) \oplus$

$\mathcal{C}(k)^\perp$  and  $V_k = s\text{-}\lim_{n \rightarrow \infty} B_k(C_k + \frac{1}{n})^{-1/2}$ . Since  $s\text{-}\lim_{k \rightarrow \infty} C_k = s\text{-}\lim_{k \rightarrow \infty} B_k = 0$  we have  $R = s\text{-}\lim_{k \rightarrow \infty} V_k V_k^*$ .

Now if  $\begin{pmatrix} D_k & E_k \\ E_k^* & F_k \end{pmatrix}$  denotes the decomposition of  $S^*AS$  relative to  $\mathcal{C}(k) \oplus \mathcal{C}(k)^\perp$ , then we have that  $F_k = C_{k+1}$  and  $E_k = S_k B_{k+1}$  as operator matrices where  $S_k = \begin{pmatrix} 0 & I & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & I \end{pmatrix}$  is  $k \times (k+1)$ . The Wold-Zasuhin part of  $S^*AS$  is the  $s$ -limit of  $W_k W_k^*$  where  $W_k = s\text{-}\lim_{n \rightarrow \infty} E_k(F_k + \frac{1}{n})^{-1/2} = s\text{-}\lim_{n \rightarrow \infty} S_k B_{k+1}(C_{k+1} + \frac{1}{n})^{-1/2} = S_k V_{k+1}$ . Since  $S^* = s\text{-}\lim_{k \rightarrow \infty} S_k$ ,  $S = s\text{-}\lim_{k \rightarrow \infty} S_k^*$ , and  $R = s\text{-}\lim_{k \rightarrow \infty} V_k V_k^*$ ,  $s\text{-}\lim_{k \rightarrow \infty} S_k V_{k+1} V_{k+1}^* S_k^* = S^*(s\text{-}\lim_{k \rightarrow \infty} V_k V_k^*)S = S^*RS$  from which the result follows.  $\square$

In the case  $n = 1$  with  $A = (A_{i,j})$  a bounded positive operator on  $l^2 \otimes \mathcal{C}$  decomposed as  $A = P + R$  it is possible to describe a canonical factorization of  $P = UU^*$  with  $U$  upper triangular. Let  $P_k$  be the positive operator such that  $H(P_k)$  is the subspace of  $H(A)$  of polynomials of degree  $k$ , each  $P_k$  has a  $k \times k$  block of non-zero entries, and  $P_k \leq P_{k+1}$  for all  $k \geq 0$ .

**Proposition 3.10.** For every  $k$ ,  $P_{k+1} - P_k$  is the  $\mathcal{C}(k)^\perp$ -minimal part of  $P_{k+1}$ .

**Proof.** Clearly,  $P_{k+1} - P_k$  agrees with  $P_{k+1}$  when restricted to  $\mathcal{C}(k)^\perp$ . Hence it is enough to show that it is the minimal such positive operator. Alternatively, we must show that for every positive operator  $R$ , with  $R \leq P_{k+1}$ ,  $Rx = 0$  for  $x \in \mathcal{C}(k)^\perp$ , we have that  $R \leq P_k$ . Since  $R \leq P_{k+1}$  we have that  $H(R) \subseteq H(P_{k+1})$  and since  $Rx = 0$  for  $x \in \mathcal{C}(k)^\perp$ ,  $H(R)$  consists of polynomials of degree at most  $k$ , so that  $H(R) \subseteq H(P_k)$ . Now for  $f \in H(R)$ ,  $\|f\|_{H(R)} \geq \|f\|_{H(P_{k+1})} = \|f\|_{H(P_k)}$  and hence  $R \leq P_k$  as desired.  $\square$

By the above result we have, applying the Cholesky result,  $P_{k+1} = \begin{pmatrix} c_1 & & & \\ \mathbf{0} & \vdots & & \\ & c_k & & \\ \mathbf{0} & c_{k+1}^{1/2} & & \end{pmatrix} \begin{pmatrix} & & & \\ c_1^* & \mathbf{0} & & \\ & \cdots & c_k^* & \mathbf{0} \\ & & & c_{k+1}^{1/2*} \end{pmatrix}$

$P_k$  where  $c_{k+1}$  is the  $(k+1, k+1)$ -entry of  $P_{k+1}$ . Denote this operator by  $C_{k+1}$  so that  $C_{k+1}C_{k+1}^* = P_{k+1} - P_k$  and  $C_{k+1}$  is only non-zero in the first  $k+1$ -entries of the  $(k+1)$ -st column. Set  $C_0 = P_0^{1/2}$ , and note that  $P = P_0 + \sum_{k=0}^{\infty} (P_{k+1} - P_k) = \sum_{k=0}^{\infty} C_k C_k^* = (\sum_{k=0}^{\infty} C_k)(\sum_{k=0}^{\infty} C_k^*)$ . Thus if we let  $U = \sum_{k=0}^{\infty} C_k$  then  $U$  is upper triangular and  $UU^* = P$ . This particular factorization  $U = (U_{i,j})$  has three additional properties:

1) the diagonal entries of  $U$  are positive;

2) for each  $k$  , if we take the decomposition of  $U$  relative to  $\mathcal{C}(k) \oplus \mathcal{C}(k)^\perp$  as

$$U = \begin{pmatrix} U_k & B_k \\ 0 & V_k \end{pmatrix} \text{ then } U_k U_k^* = \sum_{j=0}^k C_j C_j^* = P_k;$$

3) the kernel of  $U_{i,j}$  contains the kernel of  $U_{j,j}$  for all  $i$  and  $j$ .

It is not hard to see that 1) , 2) , and 3) uniquely determine the entries of  $U$  for if the first  $(k+1)$ -entries in the  $(k+1)$ -st column of  $P_k$  are  $B_0, \dots, B_k$  , then  $U_{k,k} = B_k^{1/2}$  and  $U_{i,k} B_k^{1/2} = B_i$ , since  $U_k U_k^* = P_k$ . Since by 3) ,

$$\ker(B_k) \subseteq \ker(U_{i,k}) , \text{ then } U_{i,k} = s\text{-}\lim_{n \rightarrow \infty} B_i (B_k + \frac{1}{n})^{-1/2}.$$

Note  $\mathcal{R}(U_k)$  can be identified with  $H(P_k)$  , the subspace of  $H(A)$  of polynomials of degree at most  $k$ . It is not hard to show that if  $V$  is any other upper triangular operator with  $VV^* = A$  and if  $V = \begin{pmatrix} V_k & B_k \\ 0 & W_k \end{pmatrix}$  relative to  $\mathcal{C}(k) \oplus \mathcal{C}(k)^\perp$  , then  $V_k V_k^* \leq P_k$ . Hence we call the unique factorization of  $P$  as  $UU^*$  satisfying 1) , 2) , and 3) obtained above the *maximal factorization*.

#### Section 4. Toeplitz Operators

Let  $\Phi_n \in \mathcal{B}(\mathcal{C})$  be given so that the Toeplitz operator  $T_\Phi = (\Phi_{i-j})$  is a bounded positive operator on  $l^2 \otimes \mathcal{C}$ . In this section we apply the results of the previous sections to study the reproducing kernel space  $H(T_\Phi)$ .

We set  $S = S_1 \otimes I_{\mathcal{C}}$  where  $S_1$  is the unilateral shift on  $l^2$ . We let  $M_z$  denote the operator on  $H(T_\Phi)$  defined by  $(M_z f)(z) = z f(z)$ .

**Theorem 4.1.** Let  $T_\Phi = (\Phi_{i-j})$  be a bounded positive operator on  $l^2 \otimes \mathcal{C}$ . Then the following are equivalent:

- i)  $M_z$  is bounded on  $H(T_\Phi)$ ;
- ii) if  $f(z) = \sum_{n=0}^{\infty} z^n v_n$  is in  $H(T_\Phi)$ , then the constant function  $v_0$  belongs to  $H(T_\Phi)$ ;
- iii) if  $f(z) = \sum_{n=0}^{\infty} z^n v_n$  is in  $H(T_\Phi)$  , then  $\sum_{n=0}^k z^n v_n$  is in  $H(T_\Phi)$  for each  $k$ .

**Proof.** By [2, page 383], we have that  $M_z$  is bounded if and only if

$$S T_\Phi S^* \leq c T_\Phi \text{ for some constant } c. \text{ But } S T_\Phi S^* = \begin{pmatrix} 0 & 0 \\ 0 & T_\Phi \end{pmatrix} = E T_\Phi E$$

where  $E = \begin{pmatrix} 0 & & \\ & I_{\mathcal{C}} & \\ & & \ddots \end{pmatrix}$ . Hence  $M_z$  is bounded if and only if

$E T_\Phi E \leq c T_\Phi$  for some  $c$ . But this latter condition is equivalent to  $\mathcal{R}(E T_\Phi^{1/2}) \subseteq \mathcal{R}(T_\Phi^{1/2})$  , which is equivalent to requiring that if

$\sum_{n=0}^{\infty} v_n \otimes e_n \in \mathcal{R}(T_\Phi^{1/2})$  , then  $\sum_{n=1}^{\infty} v_n \otimes e_n \in \mathcal{R}(T_\Phi^{1/2})$  . Thus,  $M_z$  is bounded if

and only if  $\sum_{n=0}^{\infty} v_n \otimes e_n \in \mathcal{R}(T_{\Phi}^{1/2})$  implies  $v_0 \otimes e_0 \in \mathcal{R}(T_{\Phi}^{1/2})$ . Applying the remark after Theorem 2.1, we see the equivalence of i) and ii).

Clearly, iii) implies ii). But if ii) holds then  $M_z$  is bounded and hence  $M_z^k$  is bounded. Applying the above argument with  $S$  replaced by  $S^k$  completes the proof that ii) implies iii).  $\square$

**Proposition 4.2.** If  $T_{\Phi} = (\Phi_{i-j})$  is a bounded positive Toeplitz operator on  $l^2 \otimes \mathcal{C}$  and  $T_{\Phi} = P + R$  is the decomposition of  $T_{\Phi}$  given by Corollary 3.5, then  $P$  and  $R$  are Toeplitz.

**Proof.** Note that  $S^*T_{\Phi}S = T_{\Phi}$ , hence by Proposition 3.9,  $S^*RS = R$  which implies that  $R$  and hence  $P$  are Toeplitz.  $\square$

Since  $S^*T_{\Phi}S = T_{\Phi}$  we have that  $S^*\mathcal{R}(T_{\Phi}^{1/2}) = \mathcal{R}(S^*T_{\Phi}^{1/2}) = \mathcal{R}(T_{\Phi}^{1/2})$  and hence  $S^*$  defines a bounded operator on  $\mathcal{R}(T_{\Phi}^{1/2})$ . Since  $S^*S = I$  on  $\mathcal{R}(T_{\Phi}^{1/2})$ ,  $S^*$  is a coisometry on  $\mathcal{R}(T_{\Phi}^{1/2})$  and  $\ker(S^*) = \{v_0 \otimes e_0 \in \mathcal{R}(T_{\Phi}^{1/2})\}$ .

**Example 4.3.** It is possible for  $T_{\Phi}$  to factor with  $M_z$  not bounded on  $H(T_{\Phi})$ . To this end let  $B_0, B_1 \in \mathcal{B}(\mathcal{C})$  and consider  $T_{\Phi} = T_{\Psi}T_{\Psi}^*$  where  $T_{\Psi} = (\Psi_{i-j})$  is the upper triangular Toeplitz operator with  $\Psi_0 = B_0, \Psi_{-1} = B_1$ , and  $\Psi_n = 0$  for  $n \neq 0, -1$ .

Assume that as subspaces of  $\mathcal{C}$ ,  $\text{ran}(B_0) \perp \text{ran}(B_1)$  and  $\ker(B_0) = \ker(B_1) = 0$ . If  $0 \neq v_0 = B_1x$  is orthogonal to  $\text{ran}(B_0)$ , then  $v_0 + zB_0x_1$  is in  $H(T_{\Phi})$  but  $v_0$  is not in  $H(T_{\Phi})$  and hence  $M_z$  is not bounded.

**Theorem 4.4.** Let  $T_{\Phi} = (\Phi_{i-j})$  be a positive Toeplitz operator on  $l^2 \otimes \mathcal{C}$  and let  $Q_n = (\Phi_{i-j})_{i,j=0}^n$ . The following are equivalent:

- i)  $M_z$  is bounded on  $H(T_{\Phi})$ ;
- ii) if  $v_0 + v_1z + \dots$  is in  $H(T_{\Phi})$ , then  $v_0 \in \mathcal{R}(Q_0^{1/2})$ ;
- iii) for each  $n$ , if  $v_0 + \dots + v_n \in H(T_{\Phi})$ , then  $v_0 \otimes e_0 + \dots + v_n \otimes e_n \in \mathcal{R}(Q_n^{1/2})$ ;
- iv) for each  $n$  the polynomials in  $H(T_{\Phi})$  of degree  $n$  all have the form  $v_0 + \dots + v_n z^n$  for  $v_0 \otimes e_0 + \dots + v_n \otimes e_n \in \mathcal{R}(Q_n^{1/2})$ .

**Proof.** This is a restatement of Theorem 4.1. Note that if we write  $T_{\Phi} = \begin{pmatrix} Q_n & B_n \\ B_n^* & C_n \end{pmatrix} = \begin{pmatrix} Q_n^{1/2} & 0 \\ X_n & D_n^{1/2} \end{pmatrix} \begin{pmatrix} Q_n^{1/2} & X_n^* \\ 0 & D_n^{1/2} \end{pmatrix}$ , then  $v_0 + v_1z + \dots$  is in  $H(T_{\Phi})$  if and only if  $\sum_{k=0}^{\infty} v_k \otimes e_k \in \mathcal{R} \left( \begin{pmatrix} Q_n^{1/2} & 0 \\ X_n & D_n \end{pmatrix} \right)$ .  $\square$

In contrast to Example 4.3 above we shall show that in the scalar case, i.e. , when  $\mathcal{C} = \mathbf{C}$  and  $T_{\phi} = (\phi_{i-j})$ ,  $\phi_n \in \mathbf{C}$ , then  $M_z$  is bounded on  $H(T_{\phi})$  if and only if  $T_{\phi}$  factors.

**Theorem 4.5** Let  $T_\Phi = (\Phi_{i-j})$  be a bounded positive Toeplitz operator on  $l^2 \otimes \mathcal{C}$ . Then the following are equivalent:

- i) The polynomials are dense in  $H(T_\Phi)$ ;
- ii)  $T_\Phi = UU^*$  with  $U$  upper triangular;
- iii)  $T_\Phi = T_G T_G^*$  with  $T_G$  upper triangular and Toeplitz.

Moreover, if  $U$  is the maximal factorization of  $T_\Phi$ , then  $U$  is Toeplitz.

**Proof.** Clearly iii) implies ii) and ii) implies i). Assume i) and let  $T_\Phi = UU^*$  be the maximal factorization. We will prove that  $U$  is Toeplitz.

Recall that  $\mathcal{R}(T_\Phi^{1/2})$  is  $S^*$ -invariant and that  $S^*$  acts as a coisometry on  $\mathcal{R}(T_\Phi^{1/2})$  with kernel  $\mathcal{R}(P_0^{1/2})$ , where  $P_k$  is the positive operator such that  $H(P_k) \subseteq H(T_\Phi)$  is the space of polynomials of degree at most  $k$ . Since  $H(P_k - P_0)$  is the subspace of  $H(T_\Phi)$  of polynomials of degree at most  $k$ , which are orthogonal to the constant polynomials,  $S^*$  acts isometrically on  $\mathcal{R}((P_k - P_0)^{1/2})$  mapping it to  $\mathcal{R}(P_{k-1}^{1/2})$ . Thus,  $S^*(P_k - P_0)S = P_{k-1}$  for  $k \geq 1$ . Since  $\mathcal{R}(S^*P_0^{1/2}) = (0)$ ,  $S^*P_0S = 0$ . Thus  $S^*P_kS = P_{k-1}$ .

Hence if the first  $(k+1)$ -entries of the  $(k+1)$ -st column of  $P_k$  are  $B_0, \dots, B_k$  then the first  $k$ -entries of the  $k$ -th column of  $P_{k-1}$  are  $B_1, \dots, B_k$ .

Thus, by the definition of the maximal  $U$  we have that

$$U_{i,k} = s\text{-}\lim_{n \rightarrow \infty} B_i(B_k + \frac{1}{n})^{-1/2}, \quad i = 0, \dots, k \text{ and}$$

$$U_{i,k-1} = s\text{-}\lim_{n \rightarrow \infty} B_{i+1}(B_{k+1} + \frac{1}{n})^{-1/2}, \quad i = 0, \dots, (k-1).$$

From this set of equations we see that  $U$  is Toeplitz.  $\square$

**Corollary 4.6.** Let  $T_\phi = (\phi_{i-j})$  be a bounded positive operator on  $l^2$ . Then  $M_z$  is bounded on  $H(T_\phi)$  if and only if  $T_\phi$  factors as  $UU^*$  with  $U$  the upper triangular matrix  $T_g^*$  for some  $g \in H^\infty$ .

**Proof.** Assume that  $M_z$  is bounded. Using Theorem 4.2, write  $\phi = \phi_1 + \psi$  where  $H(T_{\phi_1})$  is the span of the polynomials in  $H(T_\phi)$ . By Theorem 4.1,  $H(T_\phi)$  contains non-zero polynomials, so  $T_{\phi_1}$  must be nonzero. Using Theorem 4.5,  $T_{\phi_1} = T_{g_1}^* T_{g_1}$  for some  $g_1 \in H^\infty$ . Since  $\phi_1$  factors,  $\log(\phi_1)$  is integrable by the Szego alternative. But then  $\log(\phi)$  is also integrable and the Szego alternative implies  $T_\phi = T_g^* T_g$  for some  $g \in H^\infty$ .

Conversely, if  $T_\phi$  factors, then the polynomials are dense in  $H(T_\phi)$  and hence in  $\mathcal{R}(T_\phi^{1/2})$ . Since  $\mathcal{R}(T_\phi^{1/2})$  is  $S^*$ -invariant,  $e_0 \in \mathcal{R}(T_\phi^{1/2})$  and so the constant function  $1 \in H(T_\phi)$ . Clearly, condition ii) of Theorem 4.1 is met and so  $M_z$  is bounded on  $H(T_\phi)$ .  $\square$

**Remark 4.7.** Combining Theorem 4.5 and Proposition 4.2, we obtain a unique decomposition of every positive Toeplitz operator  $T_\Phi$  as  $T_\Phi = T_G T_G^* + T_\Psi$  where  $T_G = (G_{i-j})$ ,  $T_\Psi = (\Psi_{i-j})$  are Toeplitz and  $T_G$  is upper triangular. If we let  $H_n = G_n^*$ , and set  $H(e^{i\theta}) = \sum_{n=0}^{\infty} H_n e^{in\theta}$ , then  $H$  is an analytic operator-valued

function and  $\Phi(e^{i\theta}) = H(e^{i\theta})^*H(e^{i\theta}) + \Psi(e^{i\theta})$  where  $\Phi(e^{i\theta}) = \sum_{n=-\infty}^{\infty} \Phi_{-n}e^{in\theta}$  and  $\Psi(e^{i\theta}) = \sum_{n=-\infty}^{\infty} \Psi_{-n}e^{in\theta}$ .

Power's[5] obtains a similar decomposition. He requires that if  $H_1(e^{i\theta})$  is analytic and  $H_1(e^{i\theta})^*H_1(e^{i\theta}) \leq \Phi(e^{i\theta})$ , then  $H_1(e^{i\theta})H_1(e^{i\theta})^* \leq H(e^{i\theta})^*H(e^{i\theta})$ . Thus, by Theorem 3.2 iv) we see that our decomposition is the same as Power's and that by Theorem 3.2 iii) his enjoys the additional property that if  $H_2(e^{i\theta})H_2(e^{i\theta})^* \leq \Psi(e^{i\theta})$ , then  $H_2 = 0$ .

The fact that both our decompositions are the same is a bit surprising since Power's obtains his by applying the Wold-Zasuhin decomposition to the doubly infinite Toeplitz matrix  $(\Phi_{i-j})_{i,j=-\infty}^{+\infty}$ , while we have shown that our decomposition is the same as applying the Wold-Zasuhin decomposition to the operator matrix one obtains by setting all but the entries for negative  $i$  and  $j$  equal to 0.

We also obtain the following variant of the Szego alternative.

**Theorem 4.8.** Let  $T_\phi = (\phi_{i-j})$  be a positive Toeplitz operator on  $l^2$  and let

$\phi(e^{i\theta}) = \sum_{n=-\infty}^{+\infty} \phi_n e^{in\theta}$ . The following are equivalent:

- i)  $M_z$  is bounded on  $H(T_\phi)$  ;
- ii)  $\phi(e^{i\theta}) = |g(e^{i\theta})|^2$  where  $g$  is analytic ;
- iii)  $\log(\phi)$  is integrable.
- iv) 1 is in  $H(T_\phi)$ .

**Proof.** The equivalence of ii) and iii) is the Szego alternative [6]. By Theorem 4.6, i) is equivalent to  $T_\phi$  factoring as  $UU^*$  but by Theorem 4.5 this is equivalent to  $\phi$  factoring as  $gg^*$ . The equivalence of i) and iv) follows from the fact that  $1 \in \mathcal{R}(\phi_0^{1/2})$  by applying Theorem 4.4.  $\square$

**Remark 4.9.** Let  $T_\phi = (\phi_{i-j})$  be a bounded Toeplitz operator on  $l^2 \otimes \mathcal{C}$ . We conjecture that if  $M_z$  is bounded on  $H(T_\phi)$ , then  $T_\phi$  factors as  $UU^*$  with  $U$  upper triangular.

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