

ANALYTIC TRIDIAGONAL REPRODUCING KERNELS

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ABSTRACT. This paper characterizes the reproducing kernel Hilbert spaces with orthonormal bases of the form $\{(a_{n,0} + a_{n,1}z + \cdots + a_{n,J}z^J)z^n, n \geq 0\}$. The primary focus is on the tridiagonal case where $J = 1$ and how it compares to the diagonal case where $J = 0$. The question of when multiplication by z is a bounded operator is investigated and aspects of this operator are discussed. In the diagonal case M_z is a weighted unilateral shift. It is shown that in the tridiagonal case this need not be so and an example is given in which the commutant of M_z on a tridiagonal space is strikingly different from that on any diagonal space.

1. INTRODUCTION

The function $K(z, w)$ is *positive definite* (denoted $K \gg 0$) on the set $E \times E$ if for any finite collection z_1, z_2, \dots, z_n in E and any complex numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ the sum

$$\sum_{i,j=1}^n \bar{\alpha}_i \alpha_j K(z_i, z_j) \geq 0$$

with strict inequality unless all the α_i 's are zero. It is well known that if $K \gg 0$ on E , then the set of functions in z given by

$$\left\{ \sum_{j=1}^n \alpha_j K(z, w_j) : \alpha_1, \dots, \alpha_n \in \mathbb{C}, w_1, \dots, w_n \in E \right\}$$

has dense span in a Hilbert space $H(K)$ of functions on E with

$$\left\| \sum_{j=1}^n \alpha_j K(z, w_j) \right\|^2 = \sum_{i,j=1}^n \bar{\alpha}_i \alpha_j K(w_i, w_j).$$

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A fundamental property of $H(K)$ is the *Reproducing Property* which states that $f(w) = \langle f(z), K(z, w) \rangle$ for every w in E and f in $H(K)$. Thus evaluation at w is a bounded linear functional for each w in E . The function $K(z, w)$ in $H(K)$ will frequently be denoted by $K_w(z)$ or just K_w .

Conversely, it is well known that if F is a Hilbert space of functions defined on E such that evaluation at w is a bounded linear functional for each w in E , then there is a unique K defined on $E \times E$ such that $F = H(K)$. It follows from the reproducing property that $K(z, w) = \overline{K(w, z)}$. Hence if K is analytic in the first variable, then K is coanalytic in the second variable. In this case K is an *analytic* kernel. Throughout this paper, E will always be a subset of the complex plane \mathbb{C} and with few exceptions K will be an analytic kernel.

It is also well known, see N. Aronszajn[2], that if $\{f_n(z)\}$ is an orthonormal basis for a reproducing kernel Hilbert space of functions on E , then $K(z, w) = \sum_{n=0}^{\infty} f_n(z) \overline{f_n(w)}$ for all z, w in E . Moreover if the largest common domain E' of the functions $\{f_n(z)\}$ is larger than E , then the largest natural domain of $H(K)$ is given by $Dom(K) = \{z \in E' : \sum_{n=0}^{\infty} |f_n(z)|^2 < \infty\}$.

In the very well studied diagonal case $f_n(z) = z^n$ and $Dom(K)$ is always a disk. In this paper we consider only kernels where $f_n(z)$ is z^n times a polynomial of fixed degree J . Section 2 introduces these bandwidth J kernels and gives a characterization of $Dom(K)$ and $H(K)$ in this case. In sections 3 through 5 a sharper focus is put on the special case when $J = 1$. The types of spaces possible, the multipliers, and the basic matrix forms and properties of the multiplication operators are investigated. Section 6 is devoted to a single example of a tridiagonal or bandwidth 1 space in which the theory developed in the earlier sections is applied. The example is shown to have properties strikingly different from any diagonal space. In particular the commutant of multiplication by z is characterized as being quite different.

Since submission we have learned that unpublished formal versions of Theorems 2 and 3 and part of Theorem 4 were independently discovered by Sarah Ferguson during her thesis work and we would like to acknowledge her efforts. We would also like to thank Allen Schweinsberg for his many useful comments and careful reading of the manuscript.

2. BANDWIDTH J KERNELS

An analytic kernel is of finite band width J if there exists an orthonormal basis of polynomials for $H(K)$ of the form $\{f_n(z) = (a_{n,0} + a_{n,1}z + \cdots + a_{n,J}z^J)z^n, n \geq 0\}$. Kernels of bandwidth 0 are known as *diagonal* kernels and have been well studied, see A. R. Shields [4]. These spaces include the well known Hardy, Bergman, and Dirichlet spaces on the unit disk. Of particular interest in this paper are the *tridiagonal* kernels which are kernels with bandwidth 1. Our first result asserts that the natural domain $Dom(K)$ of a kernel of bandwidth J is either an open or closed disk about the origin together with at most J points not in the disk. This disk will be denoted by $\mathcal{D}(K)$.

Theorem 1. *If K is a kernel of bandwidth J , then*

1. *there exists a finite set $\Omega \subset \mathbb{C}$ of at most J points such that*

$$Dom(K) = \mathcal{D}(K) \cup \Omega \text{ where } \mathcal{D}(K) = \{z : \sum_n (|z|^n \sum_{j=0}^J |a_{n,j}|)^2 < \infty\};$$

2. *if $f = \sum_n \lambda_n f_n$ is in $H(K)$ and $z \in \mathcal{D}(K)$, then the doubly indexed series*

$$\sum_n \sum_{j=0}^J \lambda_n a_{n,j} z^{j+n} \text{ converges absolutely on } \mathcal{D}(K).$$

Proof. If $z \in \mathcal{D}(K)$, then

$$\sum_{n=0}^{\infty} |f_n(z)|^2 = \sum_n |(\sum_{j=0}^J a_{n,j} z^j) z^n|^2 \leq \{max|z|, 1\}^{2J} \sum_n (|z|^n \sum_{j=0}^J |a_{n,j}|)^2 < \infty.$$

Since $Dom(K) = \{z \in \mathbb{C} : \sum |f_n(z)|^2 < \infty\}$, this shows $z \in Dom(K)$. It remains to show that $Dom(K)$ can have at most J additional points.

To this end assume that z_1, \dots, z_{J+1} are distinct points in $Dom(K)$ with $|z_1| \leq |z_2| \leq \cdots \leq |z_{J+1}|$ and $z_1 \notin \mathcal{D}(K)$. By assumption $\sum_n |f_n(z_k)|^2 < \infty$ for each $k = 1, \dots, J+1$. Since $|z_1| \leq |z_k|$ for each k ,

$$\sum_n |(\sum_{j=0}^J a_{n,j} z_k^j) z_1^n|^2 \leq \sum_n (|(\sum_{j=0}^J a_{n,j} z_k^j)| |z_1^n|)^2 < \infty.$$

Thus, $\sum_n |\sum_{k=1}^{J+1} (\sum_{j=0}^J \alpha_k a_{n,j} z_k^j) z_1^n|^2 < \infty$ for any constants $\alpha_1, \alpha_2, \dots, \alpha_{J+1}$. Letting

$\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_{J+1})^T$, $\vec{a}_n = (\overline{a_{n,0}}, \overline{a_{n,1}}, \dots, \overline{a_{n,J}})^T$ in \mathbb{C}^{J+1} , and $V = \{v_{j,k}\}$ the Vandermonde matrix with $v_{j,k} = z_k^j$ one has $\sum_{k=1}^{J+1} \sum_{j=0}^J \alpha_k a_{n,j} z_k^j = \langle V \vec{\alpha}, \vec{a}_n \rangle$. Now

V is known to be invertible so long as the points z_1, z_2, \dots, z_{J+1} are distinct. Fix $j \in \{0, 1, \dots, J\}$ and choose $\vec{\alpha} \in \mathbb{C}^{J+1}$ so that $V\vec{\alpha} = \vec{e}_j = (0, 0, \dots, 1, 0, \dots, 0)^T$. Hence $\langle V\vec{\alpha}, \vec{a}_n \rangle = a_{n,j}$ and therefore $\sum_n (|a_{n,j}| |z_1|^n)^2 < \infty$.

Thus $\sum_n ((\sum_{j=0}^J |a_{n,j}|) |z_1|^n)^2 < \infty$ showing $z_1 \in \mathcal{D}(K)$. This contradicts our original assumptions and proves the first part of the theorem.

Now assume $f = \sum_n \lambda_n f_n$ converges in $H(K)$ and $z \in \mathcal{D}(K)$. By the first part of the theorem $\sum_n (\sum_{j=0}^J |a_{n,j}| |z|^n)^2 = C < \infty$. Hence both the sequences $\{\lambda_n\}_{n=0}^\infty$ and $\{\sum_{j=0}^J |a_{n,j}| |z|^n\}_{n=0}^\infty$ are in l^2 . Let $M = \max\{1, |z|^J\}$. By the Cauchy-Schwarz inequality $\sum_n \sum_{j=0}^J |\lambda_n a_{n,j} z^{j+n}| \leq M \|\{\lambda_n\}\|_2 \|\{\sum_{j=0}^J |a_{n,j}| |z|^n\}\|_2 = MC \|f\|_{H(K)}$. \square

An important consequence of the above theorem is that on any closed disk about the origin in $Dom(K)$ the series $\sum_n \lambda_n f_n(z)$ can be rearranged to converge coefficient wise to the Taylor series of $f(z)$ about the origin. Thus if $f_n(z) = (a_n + b_n z)z^n$, the series $\sum \lambda_n (a_n + b_n z)z^n$ can be rearranged to give the Taylor series $\sum (\lambda_n a_n + \lambda_{n-1} b_{n-1})z^n$.

For a tridiagonal kernel ($J = 1$) the theorem states that $Dom(K)$ is either a closed or an open disk about 0 plus possibly a single additional point z_0 . Before proceeding a few examples will be given for illustration. In the examples below the basis functions are assumed of the form $f_n(z) = (a_n + b_n z)z^n$.

- Example 1.**
1. If $a_n = 1$ and $b_n = 0$ for each n , then $Dom(K) = \mathbb{D}$ where \mathbb{D} denotes the unit disk and $H(K)$ is the Hardy space $H^2(\mathbb{D})$.
 2. If $a_n = \sqrt{n+1}$ and $b_n = 0$ for each n , then $Dom(K) = \mathbb{D}$ and $H(K)$ is the Bergman space.
 3. If $a_n = 1/\sqrt{n+1}$ and $b_n = 0$ for each n , then $Dom(K) = \mathbb{D}$ and $H(K)$ is the Dirichlet space.

Example 2. If $a_n = 1$ and $b_n = \frac{-1}{2}$ for each n , then $Dom(K) = \mathbb{D} \cup \{2\}$. Here the function $K_2(z) = K(z, 2)$ has norm 0 and $H(K)$ is just a renorming of the Hardy space $H^2(\mathbb{D})$.

Example 3. Let $a_n = 1$ and $b_n = \frac{-1}{2}$ for n even; let $a_n = (\frac{1}{3})^n$ and $b_n = 0$ for n odd. Then $Dom(K) = \mathbb{D} \cup \{2\}$, but $\|K_2\| > 0$. It will be shown later that $H(K)$ is not a renorming of a diagonal space and that multiplication by z is unbounded.

Example 4. If $a_n = 1$, $b_n = \frac{n+1}{n+2}$ then $Dom(K) = \mathbb{D} \cup \{-1\}$. This example will be extensively investigated later. It will be shown that $H(K)$ is not a diagonal space, multiplication by z is bounded, and that the commutant of multiplication by z is quite interesting. Changing to $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n+2}$ here results in $H(K)$ being the same in most respects but $Dom(K) = \overline{\mathbb{D}}$.

The remainder of this paper is devoted to characterizing the spaces $H(K)$ when K is a tridiagonal kernel and investigating the behavior of various multiplication operators, particularly multiplication by z . In particular we will determine when a tridiagonal space is a renorming of a diagonal space and illustrate the role that z_0 can play when present in $Dom(K)$.

Before proceeding it is worthwhile to recall a few facts that will both illuminate why the terms *diagonal*, *tridiagonal*, and *finite band width* are associated with the kernels we are studying and that will also facilitate our computations.

If $K(z, w)$ is an analytic reproducing kernel in a neighborhood of $(0, 0)$, then K has a power series expansion $\sum a_{ij} z^i \bar{w}^j$ about $(0, 0)$ and the reproducing kernel Hilbert space obtained by restricting K to the connected domain containing $(0, 0)$ will be denoted by $H(A)$. In Adams, McGuire, Paulsen [1] it is shown that the coefficient matrix $A = [a_{i,j}]$ is a positive definite matrix in l^2 and that $H(A)$ is isomorphic to $R(B)$ where $A = BB^*$ and $R(B)$ is the range space of B equipped with the norm $\|B\vec{x}\| = \|\vec{y}\|$ where \vec{y} is the unique vector in $(ker B)^\perp$ such that $B\vec{y} = B\vec{x}$. The isomorphism associates the function $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$ with the vector $(\alpha_0, \alpha_1, \dots)^T$ in $R(B)$. By a dilation the matrix A may be assumed bounded and can always be factored via the Cholesky decomposition into the form $A = LL^*$ where L is lower triangular. Moreover if the j, j th entry of L is zero, then the j th column of L is zero. In this case the nonzero columns of L form an orthonormal basis for $R(B)$. Hence one may assume without loss of generality that $A = LL^*$ where L is lower triangular, the j, j th entry of L is nonzero, and L has trivial kernel. Thus a *diagonal* kernel corresponds to the matrices A and L being diagonal and an orthonormal basis consisting of powers of z . For a *tridiagonal* kernel the matrix A

is tridiagonal and L has nonzero entries only on the diagonal and first subdiagonal. Similarly J finite band width means A has a band width of $2J + 1$ and L has nonzero entries only on the diagonal and first J subdiagonals.

The functions in $H(K)$ have as their natural domain the union of $\mathcal{D}(K)$ together with possibly finitely many points. Restricted to the interior of $\mathcal{D}(K)$ the functions are seen to be analytic. By $H(LL^*)$ we will mean the restriction of $H(K)$ to $\mathcal{D}(K)$ with the original norm. By Theorem 1 the functions in $H(K)$ are determined by their values on $\mathcal{D}(K)$ and hence there is no "real loss" in this restriction. The purpose of the restriction is that in the presence of a point z_0 outside $\mathcal{D}(K)$ there is no reason to expect that a multiplication operator such as multiplication by z will behave as "multiplication by z " as the series representations of functions at the point z_0 are not absolutely convergent and not amenable to rearrangements (particularly when multiplied by z). We will illustrate this later by showing multiplication by z , denoted by M_z , is bounded on the space of Example 4 but M_z does not multiply by -1 at the point -1 .

3. TRIDIAGONAL KERNEL SPACES

Henceforth we will assume $A = LL^*$ is tridiagonal with $L = \begin{pmatrix} a_0 & 0 & 0 & \cdots \\ b_0 & a_1 & 0 & \cdots \\ 0 & b_1 & a_2 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}$

bounded, having trivial kernel, and $a_n \neq 0$ for all n . The restriction of $H(K)$ to $\mathcal{D}(K)$ will be denoted by $H(LL^*)$. As mentioned before $H(LL^*)$ has exactly the same orthonormal basis, contains the same functions, and is exactly the same as $H(K)$ with the exception of the possible single point z_0 in its domain. The next results characterize when $H(LL^*)$ contains the polynomials and when it is a renorming of a diagonal space.

Theorem 2. *The polynomials are contained in $H(LL^*)$ if and only if for each n the sequence $\{1, \frac{b_n}{a_{n+1}}, \frac{b_n b_{n+1}}{a_{n+1} a_{n+2}}, \frac{b_n b_{n+1} b_{n+2}}{a_{n+1} a_{n+2} a_{n+3}}, \dots\}$ is square summable.*

Proof. Note $z^n \in H(LL^*)$ if and only if for some square summable sequence $\{\lambda_l\}$, $z^n = \sum_{l=0}^{\infty} \lambda_l (a_l + b_l z) z^l$. By Theorem 1, $\lambda_l a_l + \lambda_{l-1} b_{l-1} = \begin{cases} 1 & \text{if } l = n \\ 0 & \text{if } l \neq n \end{cases}$. Note

$1 \in H(LL^*)$ implies $a_0 \neq 0$ and in like manner $a_l \neq 0$ each l . This shows that $\lambda_0, \dots, \lambda_{n-1} = 0$ and $\lambda_n = \frac{1}{a_n}$. Thus $\lambda_{n+1}a_{n+1} + \lambda_n b_n = 0$ or $\lambda_{n+1} = -\frac{1}{a_n} \frac{b_n}{a_{n+1}}$. Solving iteratively shows $\lambda_{n+k} = \frac{(-1)^k}{a_n} \frac{b_n \cdots b_{n+k-1}}{a_{n+1} \cdots a_{n+k}}$. Since $\frac{1}{a_n}$ is a constant for fixed n it does not affect the square summability of the sequence and may be dropped. \square

As $H(LL^*)$ is a space of functions naturally defined on a disk and a space spanned by polynomials, it is natural to ask if it is functionally the same as a space where the powers of z are mutually orthogonal. Since such a space would arise as $H(DD^*)$ where D is a diagonal matrix, we are asking when $H(LL^*) = H(DD^*)$ as a set of functions. This is equivalent to $\text{Range}(L) = \text{Range}(D)$ where $\text{Range}(L)$ is the range space, see Adams, McGuire, Paulsen [1]. By R. G. Douglas [3], $\text{Range}(L) = \text{Range}(D)$ is equivalent to the existence of bounded matrices C_1 and C_2 such that $LC_1 = D$ and $DC_2 = L$. Thus $LC_1C_2 = L$. Since L is one-to-one, $C_1C_2 = I$. Similarly $C_2C_1 = I$ and thus $C_2 = C_1^{-1}$. This leads to the following characterization.

Theorem 3. If $L = \begin{pmatrix} a_0 & 0 & \cdots \\ b_0 & a_1 & \ddots \\ 0 & b_1 & \ddots \\ \vdots & \ddots & \ddots \end{pmatrix}$, then $H(LL^*) = H(DD^*)$ for some

diagonal matrix D if and only if $\limsup_{j \rightarrow \infty} \sup_n \left(\prod_{k=0}^{j-1} \left| \frac{b_{n+k}}{a_{n+k+1}} \right| \right)^{1/j} < 1$ and $\sup \left| \frac{b_n}{a_{n+1}} \right| < \infty$.

If $H(LL^*) = H(DD^*)$, then D may be taken to equal $\begin{pmatrix} a_0 & 0 & & \\ 0 & a_1 & \ddots & \\ & \ddots & \ddots & \\ & & & \ddots \end{pmatrix}$.

Proof. First assume $LC = D$ where C is invertible and $D = \begin{pmatrix} d_0 & & & \\ & d_1 & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix}$. A

straightforward computation shows C must be lower triangular with main diagonal entries $\frac{d_0}{a_0}, \frac{d_1}{a_1}, \dots$. Thus $\sup_n \left| \frac{d_n}{a_n} \right| < \infty$. Hence the entries on the main diagonal of C^{-1} are $\frac{a_0}{d_0}, \frac{a_1}{d_1}, \dots$ implying $\sup_n \left| \frac{a_n}{d_n} \right| < \infty$. This shows that $\text{Range}(DD^*) = \text{Range}(\text{Diag}(a_0, a_1, \dots) \text{Diag}(a_0, a_1, \dots)^*)$ and therefore that D

may be assumed to equal $Diag(a_0, a_1, \dots)$. With this assumption a short com-

putation shows that $C = \begin{pmatrix} 1 & 0 & 0 & \dots \\ -\frac{b_0}{a_1} & 1 & 0 & \ddots \\ \frac{b_0 b_1}{a_1 a_2} & -\frac{b_1}{a_2} & 1 & \ddots \\ -\frac{b_0 b_1 b_2}{a_1 a_2 a_3} & \frac{b_1 b_2}{a_2 a_3} & -\frac{b_2}{a_3} & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} = (1 + S_\omega)^{-1}$ where

S_ω is the shift which maps \vec{e}_n to $\frac{b_n}{a_{n+1}}\vec{e}_{n+1}$. The matrix C is shown in A. R. Shields [4] to be invertible precisely when the spectral radius $\lim_{j \rightarrow \infty} \|S_\omega^j\|^{1/j} < 1$.

But $\|S_\omega^j\|^{1/j} = \sup_n \left(\prod_{k=0}^{j-1} \left| \frac{b_{n+k}}{a_{n+k+1}} \right| \right)^{1/j}$. To prove the other direction simply note that $\lim_{j \rightarrow \infty} \|S_\omega^j\|^{1/j} < 1$ implies that C is invertible and therefore that $H(LL^*) = H(DD^*)$ where $D = Diag(a_0, a_1, \dots)$. \square

4. MULTIPLICATION BY z

Since $H(LL^*)$ is a space of analytic functions it is natural to ask if multiplication by the independent variable z is a bounded operator. Let M_z denote the operator and \widehat{M}_z its formal matrix with respect to the basis $\{f_n(z) = (a_n + b_n z)z^n, n \geq 0\}$. Note $M_z f_n = a_n z^{n+1} + b_n z^{n+2} = \frac{a_n}{a_{n+1}} f_{n+1} + \left(\frac{b_n}{a_{n+2}} - \frac{a_n}{a_{n+1}} \frac{b_{n+1}}{a_{n+2}} \right) a_{n+2} z^{n+2}$. Letting $c_n = \frac{b_n}{a_{n+2}} - \frac{a_n}{a_{n+1}} \frac{b_{n+1}}{a_{n+2}}$ and calculating as in Theorem 2, \widehat{M}_z is revealed to have the form

$$\widehat{M}_z = \begin{pmatrix} 0 & 0 & 0 & \dots \\ \frac{a_0}{a_1} & 0 & 0 & \ddots \\ c_0 & \frac{a_1}{a_2} & 0 & \ddots \\ \frac{-c_0 b_2}{a_3} & c_1 & \frac{a_2}{a_3} & \ddots \\ \frac{c_0 b_2 b_3}{a_3 a_4} & \frac{-c_1 b_3}{a_4} & c_2 & \ddots \\ \frac{-c_0 b_2 b_3 b_4}{a_3 a_4 a_5} & \frac{c_1 b_3 b_4}{a_4 a_5} & \frac{-c_2 b_4}{a_5} & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

Before proceeding two things are worth noting. First the operator M_z is bounded if and only if \widehat{M}_z is the matrix of a bounded operator. Second \widehat{M}_z is formally

$$S_\psi + S^2(I - S_\omega + S_\omega^2 - S_\omega^3 + \dots)D = S_\psi + S^2(I + S_\omega)^{-1}D$$

where $S_\psi = \begin{pmatrix} 0 & 0 & \cdots \\ \frac{a_0}{a_1} & 0 & \ddots \\ 0 & \frac{a_1}{a_2} & \ddots \\ \vdots & \ddots & \ddots \end{pmatrix}$ and $S_\omega = \begin{pmatrix} 0 & 0 & \cdots \\ \frac{b_2}{a_3} & 0 & \ddots \\ 0 & \frac{b_3}{a_4} & \ddots \\ \vdots & \ddots & \ddots \end{pmatrix}$ are weighted shifts, S is

the usual unilateral shift, and $D = \text{Diag}(c_0, c_1, \dots)$ is a diagonal matrix. This second fact will not be made use of but is worth noting as another form of M_z since it illustrates how the operator differs algebraically from a weighted shift and gives a clue that the operator algebraic structures of M_z such as the commutant are more complicated. For the remainder of this paper M_z and \widehat{M}_z will be used interchangeably. A cursory glance at the form of the matrix \widehat{M}_z is enough to recognize that the columns are not unrelated to the sequences in Theorem 2. Our next result explores this relationship.

Theorem 4. *If M_z is bounded on $H(LL^*)$, then*

1. $|\frac{a_n}{a_{n+1}}|$ is bounded for all n ;
2. $|c_n|$ is bounded for all n where $c_n = \frac{1}{a_{n+2}}(b_n - \frac{a_n}{a_{n+1}}b_{n+1})$;
3. if $c_n \neq 0$ for some n , then $H(LL^*)$ contains the polynomials.
4. if $c_n = 0$ for all n , then $H(LL^*) = (1 + \alpha z)H(DD^*)$ where $D = \text{Diag}[a_0, a_1, \dots]$ and α is the constant ratio $\frac{b_n}{a_n}$. In this case M_z is a weighted shift with weight sequence $\{\frac{a_n}{a_{n+1}}\}_{n=0}^\infty$.

Proof. Since $\frac{a_n}{a_{n+1}}$ and c_n are the diagonal and subdiagonal entries of the matrix \widehat{M}_z , parts one and two of the theorem are immediate. For part three note that if $c_n \neq 0$, then in order that the n th column of \widehat{M}_z be in l^2 the sequence $\{\frac{b_{n+2}}{a_{n+3}}, \frac{b_{n+2}b_{n+3}}{a_{n+3}a_{n+4}}, \dots\}$ must be square summable. Theorem 2 now implies the monomial z^{n+2} is in $H(LL^*)$. Since M_z is bounded, this implies z^m is in $H(LL^*)$ for all $m \geq n + 2$. If $m < n + 2$, then factoring $\frac{b_m \cdots b_{n+1}}{a_{m+1} \cdots a_n}$ from the tail of the sequence in Theorem 2, one obtains the sequence $\{\frac{b_{n+2}}{a_{n+3}}, \frac{b_{n+2}b_{n+3}}{a_{n+3}a_{n+4}}, \dots\}$. Since this sequence is square summable, the condition of Theorem 2 is met. Hence all the powers of z are in $H(LL^*)$ and $H(LL^*)$ contains the polynomials. Part four follows from noting that if $c_n = 0$ for all n , then $\frac{b_n}{a_n} = \frac{b_{n+1}}{a_{n+1}}$ for all n and hence $f_n(z) = (1 + \frac{b_n}{a_n}z)a_n z^n = (1 + \alpha z)a_n z^n$. Thus $H(LL^*)$ is $(1 + \alpha z)$ times the diagonal space which has $\{a_n z^n\}$ as an orthonormal basis. \square

Using Theorem 4 it is now easy to verify that multiplication by z is unbounded in Example 3. Because of the close connection between the columns of \widehat{M}_z and the sequences in Theorem 2 one might hope the converse of Theorem 4 could hold true. The following example shows the converse is false.

Example 5. Let $a_n = 1$ for each n and

$$b_n = \begin{cases} 1, & \text{if } 10^m + 1 \leq n \leq 10^m + 10^{m-1} \text{ for any } m \\ 0, & \text{otherwise} \end{cases}$$

As all of the sequences in Theorem 2 are finite it is clear that $H(LL^*)$ contains the polynomials. It is also clear that $\sup \frac{a_n}{a_{n+1}} = 1 < \infty$ and that for each n , $|c_n| = |b_n - b_{n+1}| \leq 1$. Note $c_{10^m} = \frac{1}{1}(0 - \frac{1}{1}) = -1$ and that the 10^m th column of \widehat{M}_z has 10^{m-1} elements of absolute value 1. Hence $\|M_z f_n\|$ is unbounded and thus M_z is unbounded.

The next two examples illustrate why $c_n \neq 0$ in Theorem 4 part (3). In both examples $c_n = 0$.

Example 6. Let $a_n = \frac{1}{2^n}$ and $b_n = \frac{1}{2^{n+1}}$ for $n \geq 0$. Note $\widehat{M}_z = \begin{pmatrix} 0 & 0 & \cdots \\ 2 & 0 & \ddots \\ 0 & 2 & \ddots \\ \vdots & \ddots & \ddots \end{pmatrix}$

and hence M_z is bounded. However $\{1, \frac{b_0}{a_1}, \frac{b_0 b_1}{a_1 a_2}, \dots\} = \{1, 1, 1, \dots\}$ is not square summable. By Theorem 2 the constant function 1 is not in $H(LL^*)$. Also Theorem 3 shows that $H(LL^*)$ is not a diagonal space.

Example 7. Let $a_n = 2(\frac{2}{3})^n$ and $b_n = (\frac{2}{3})^n$ for $n \geq 0$. Note $\widehat{M}_z = \begin{pmatrix} 0 & 0 & \cdots \\ \frac{3}{2} & 0 & \ddots \\ 0 & \frac{3}{2} & \ddots \\ \vdots & \ddots & \ddots \end{pmatrix}$

and hence M_z is bounded. Moreover $\{1, \frac{b_0}{a_1}, \frac{b_0 b_1}{a_1 a_2}, \dots\} = \{1, \frac{3}{4}, (\frac{3}{4})^2, \dots\}$ is square summable and hence 1 is in $H(LL^*)$. Thus $H(LL^*)$ contains all the polynomials. In fact Theorem 3 shows $H(LL^*)$ is a diagonal space.

The next result points toward a converse of Theorem 4.

Theorem 5. *If $\sup \left| \frac{a_n}{a_{n+1}} \right| < \infty$ and $\limsup \left| \frac{b_n}{a_{n+1}} \right| < 1$, then M_z is bounded.*

Proof. Recall $\widehat{M}_z = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ \frac{a_0}{a_1} & 0 & 0 & \ddots \\ c_0 & \frac{a_1}{a_2} & 0 & \ddots \\ \frac{-c_0 b_2}{a_3} & c_1 & \frac{a_2}{a_3} & \ddots \\ \frac{c_0 b_2 b_3}{a_3 a_4} & \frac{-c_1 b_3}{a_4} & c_2 & \ddots \\ \frac{-c_0 b_2 b_3 b_4}{a_3 a_4 a_5} & \frac{c_1 b_3 b_4}{a_4 a_5} & \frac{-c_2 b_4}{a_5} & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$ and let M_j be the matrix

obtained from \widehat{M}_z by setting entries off the j th subdiagonal to 0. Note $\widehat{M}_z = \sum_{j=0}^{\infty} M_j$ and that by assumption $\|M_1\| = \sup \left| \frac{a_n}{a_{n+1}} \right| < \infty$. Since $\limsup \left| \frac{b_n}{a_{n+1}} \right| < 1$ there exists $r < 1$ and n_0 such that $\left| \frac{b_n}{a_{n+1}} \right| < r$ for $n \geq n_0$. Also $c_n = \frac{b_n}{a_{n+2}} - \frac{a_n}{a_{n+1}} \frac{b_{n+1}}{a_{n+2}} = \frac{b_n}{a_{n+1}} \frac{a_{n+1}}{a_{n+2}} - \frac{a_n}{a_{n+1}} \frac{b_{n+1}}{a_{n+2}}$ is bounded. Let $K = \sup_n \left\{ \left| \frac{b_n}{a_{n+1}} \right|, |c_n| \right\}$ and note

$$\|M_j\| \leq \begin{cases} K^{j-1}, & \text{if } 2 \leq j \leq n_0 \\ K^{n_0} r^{j-n_0}, & \text{if } j > n_0 \end{cases}.$$

Hence $\sum \|M_j\| < \infty$ and \widehat{M}_z is bounded. \square

Before closing this section we turn our attention to the matrix form of M_{z^p} with an end to obtaining an estimate of the spectral radius of M_z . First note that if

$$c_{n,p} = \frac{b_n}{a_{n+p+1}} - \frac{a_n}{a_{n+p}} \frac{b_{n+p}}{a_{n+p+1}}, \text{ then}$$

$$\begin{aligned} M_{z^p}(f_n) &= \frac{a_n}{a_{n+p}} f_{n+p} + (b_n - \frac{a_n}{a_{n+p}} b_{n+p}) z^{n+p+1} = \\ &= \frac{a_n}{a_{n+p}} f_{n+p} + c_{n,p} (f_{n+p+1} - \frac{b_{n+p+1}}{a_{n+p+2}} f_{n+p+2} + \frac{b_{n+p+1} b_{n+p+2}}{a_{n+p+2} a_{n+p+3}} f_{n+p+3} - \cdots). \end{aligned}$$

Proof. Note that $M_{z^p} = A_p + B_p$ where $A_p = \begin{pmatrix} p \text{ rows of zeros} \\ \dots \\ \frac{a_0}{a_p} & 0 & \dots \\ 0 & \frac{a_1}{a_{p+1}} & \ddots \\ \vdots & \ddots & \ddots \end{pmatrix}$ and

$$B_p = \begin{pmatrix} p \text{ rows of zeros} \\ \dots \\ 0 & 0 & 0 & \dots \\ c_{0,p} & 0 & 0 & \dots \\ \frac{-c_{0,p}b_{p+1}}{a_{p+2}} & c_{1,p} & 0 & \ddots \\ \frac{c_{0,p}b_{p+1}b_{p+2}}{a_{p+2}a_{p+3}} & \frac{-c_{1,p}b_{p+2}}{a_{p+3}} & c_{2,p} & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} .$$

For $p \geq 1$ the matrix B_p is equal to $B_1 S^{p-1} D_p$ where S is the usual unilateral shift and D_p is the diagonal matrix

$Diag[\frac{c_{0,p}}{c_{p-1,1}}, \frac{c_{1,p}}{c_{p,1}}, \frac{c_{2,p}}{c_{p+1,1}}, \dots]$. By Lemma 1, $\|B_1\| = \|M_z - A_1\| \leq 2\|M_z\|$. Hence

$$\|B_p\| \leq \|B_1\| \sup_{n \geq 0} \left| \frac{c_{n,p}}{c_{n+p-1,1}} \right| \leq 2\|M_z\| \sup_{n \geq 0} \left| \frac{c_{n,p}}{c_{n+p-1,1}} \right|.$$

Also

$$\|A_p\| \leq \sup_{n \geq 0} \left| \frac{a_{n+1}}{a_{n+p}} \right| \|A_1\| \leq \sup_{n \geq 0} \left| \frac{a_{n+1}}{a_{n+p}} \right| \|M_z\|.$$

Thus, $\|M_z^p\| \leq \left(\sup_{n \geq 0} \left| \frac{a_{n+1}}{a_{n+p}} \right| + 2 \sup_{n \geq 0} \left| \frac{c_{n,p}}{c_{n+p-1,1}} \right| \right) \|M_z\|$. Hence $\limsup_{p \rightarrow \infty} \|M_z^p\|^{\frac{1}{p}}$ gives the result since $\limsup_{p \rightarrow \infty} \|M_z\|^{\frac{1}{p}} = 1$. \square

5. THE MATRIX FORM OF M_ϕ

As before we assume L is bounded with trivial kernel, the n, n th entry of L is nonzero, and $f_n(z) = (a_n + b_n z)z^n$. In this case $H(LL^*)$ consists of functions of the form $\sum_{n=0}^{\infty} \lambda_n f_n(z)$ such that $\sum_{n=0}^{\infty} |\lambda_n|^2 < \infty$. Assume $\phi(z) = \sum_{n=0}^{\infty} \phi_n z^n$ is an analytic function in $\mathcal{D}(K)$ such that for each n , $g_n(z) = \phi(z)f_n(z)$ is in $H(LL^*)$. Note that $\phi(z)$ is not assumed to be in $H(LL^*)$. Since $g_n \in H(LL^*)$,

$$g_n(z) = \sum_{m=0}^{\infty} g_{n,m} f_m(z) = \left(\sum_{k=0}^{\infty} \phi_k z^k \right) (a_n + b_n z) z^n.$$

Note the lowest power of z on the right hand side is z^n and hence $g_{n,m} = 0$ for $m < n$ as the functions in $H(LL^*)$ are determined by their power series expansions.

The remaining coefficients $g_{n,m}$ can be obtained recursively from the equation

$$\sum_{m=n}^{\infty} (g_{n,m} a_m z^m + g_{n,m} b_m z^{m+1}) = \sum_{k=0}^{\infty} (a_n \phi_k z^{n+k} + b_n \phi_k z^{n+k+1}).$$

For example $g_{n,n} a_n = a_n \phi_0$ implies $g_{n,n} = \phi_0$ and $g_{n,n+1} a_{n+1} + g_{n,n} b_n = a_n \phi_1 + b_n \phi_0$ implies $g_{n,n+1} = \frac{a_n}{a_{n+1}} \phi_1$. Proceeding $g_{n,n+2} = \frac{a_n}{a_{n+2}} \phi_2 + c_{n,1} \phi_1$ and one can then show that for $J \geq 3$,

$$g_{n,n+J} = \frac{a_n}{a_{n+J}} \phi_J + c_{n,J-1} \phi_{J-1} + \sum_{j=1}^{J-2} c_{n,j} \phi_j \prod_{p=j}^{J-2} \left(\frac{-b_{n+p+1}}{a_{n+p+2}} \right).$$

Thus with respect to the basis $\{f_n(z)\}$ the formal matrix for the operator M_ϕ of multiplication by ϕ is given by $M_\phi = [g_{m,n}]$ where $g_{m,n}$ is the (n, m) th entry.

This realization of the matrix of M_ϕ can also be obtained from the form of $M_{z^p} = A_p + B_p$ of section 4. Recall A_p is a matrix whose only nonzero entries are $\frac{a_0}{a_p}, \frac{a_1}{a_{p+1}}, \dots$ on the $(p+1)$ th subdiagonal and $B_p = B_1 S^{p-1} D_p$ where S is the unilateral shift and $D_p = \text{Diag}[\frac{c_{0,p}}{c_{p-1,1}}, \frac{c_{1,p}}{c_{p,1}}, \frac{c_{2,p}}{c_{p+1,1}}, \dots]$. Since A_p can also be viewed as S_ψ^p where S_ψ is the unilateral weighted shift with the weight sequence $\{\frac{a_0}{a_1}, \frac{a_1}{a_2}, \frac{a_2}{a_3}, \dots\}$ formally

$$M_\phi = \sum_{p=0}^{\infty} \phi_p S_\psi^p + \sum_{p=1}^{\infty} \phi_p B_1 S^{p-1} D_p = \phi(S_\psi) + B_1 \sum_{p=1}^{\infty} \phi_p S^{p-1} D_p$$

and the entries agree with those obtained above.

Viewed in this manner the matrix of M_ϕ is suggestive of a realization of the operator M_ϕ as a sum of $\phi(S_\psi)$ and B_1 times some type of an operator weighted derivative of $\phi(S_\psi)$. In our next section we will use the above formulation of M_ϕ to work out the commutant of M_z in Example 4 and lend some weight to this strangely different suggested form of M_ϕ .

6. AN INTERESTING EXAMPLE

The *commutant* of an operator A is the set of all operators B such that $AB = BA$. By Theorem 4, if M_z is bounded either $H(LL^*)$ contains the constant function 1 or $H(LL^*)$ is a linear function times a diagonal space. In either case it is straightforward to show that if B is in the commutant of M_z , then $B = M_\phi$ for

some function ϕ analytic on $\mathcal{D}(K)$. If 1 is in $H(LL^*)$, then $\phi = B(1)$ is in $H(LL^*)$. If 1 is not in $H(LL^*)$, then $\phi = \frac{B(f_0)}{f_0}$ is not in $H(LL^*)$. In the case of the Hardy space the commutant of M_z is the set of all *Toeplitz* matrices T_ϕ where the symbol ϕ is in $H^\infty(\mathbb{D})$, the set of bounded analytic functions on \mathbb{D} . Below we will show that the commutant of M_z in the case of Example 4 is quite unlike any diagonal case.

Recall that in Example 4, $a_n = 1$, $b_n = \frac{n+1}{n+2}$, and $Dom(K) = \mathbb{D} \cup \{-1\}$.

Theorem 7. *If $f_n(z) = (1 + \frac{n+1}{n+2}z)z^n$ for $n \geq 0$, then*

1. $H(LL^*)$ is not a diagonal space;
2. $H(LL^*)$ contains the polynomials;
3. M_z is bounded and a Hilbert-Schmidt perturbation of the unilateral shift;
4. the spectrum of M_z is the closed unit disk;

Proof. Theorem 3 implies $H(LL^*)$ is not a diagonal space since

$\limsup_{j \rightarrow \infty} \sup_n \left(\prod_{k=0}^{j-1} \left| \frac{b_{n+k}}{a_{n+k+1}} \right| \right)^{1/j} = \limsup_{j \rightarrow \infty} \sup_n \left(\frac{n+1}{n+j+1} \right)^{1/j} = 1$. Since the sequence $\{\frac{n+1}{n+j+1}\}$ is square summable Theorem 2 shows that $H(LL^*)$ contains the polynomials. However since $\limsup \frac{b_n}{a_{n+1}} = 1$, Theorem 5 does not apply and we must show directly that M_z is bounded. To this end note that $c_{n,p} = b_n - b_{n+p} = \frac{n+1}{n+2} - \frac{n+p+1}{n+p+2} = \frac{-p}{(n+2)(n+p+2)}$. Thus the matrix form of M_z is

$$\widehat{M}_z = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \ddots \\ \frac{-1}{2 \cdot 3} & 1 & 0 & \ddots \\ \frac{1}{2 \cdot 4} & \frac{-1}{3 \cdot 4} & 1 & \ddots \\ \frac{-1}{2 \cdot 5} & \frac{1}{3 \cdot 5} & \frac{-1}{4 \cdot 5} & \ddots \\ \frac{1}{2 \cdot 6} & \frac{-1}{3 \cdot 6} & \frac{1}{4 \cdot 6} & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Since the doubly indexed series $\sum (\frac{1}{m \cdot n})^2$ converges M_z is seen to be not only bounded but a Hilbert-Schmidt perturbation of the unilateral shift S .

Applying Theorem 6 the spectral radius of M_z is seen to be bounded by

$$\begin{aligned}\alpha &= \limsup_{p \rightarrow \infty} \left(\sup_{n \geq 0} \left| \frac{a_{n+1}}{a_{n+p}} \right| + 2 \sup_{n \geq 0} \left| \frac{c_{n,p}}{c_{n+p-1,1}} \right| \right)^{\frac{1}{p}} \\ &= \limsup_{p \rightarrow \infty} \left(1 + 2 \sup_{n \geq 0} \frac{p(n+p+1)}{n+2} \right)^{\frac{1}{p}} \\ &= \limsup_{p \rightarrow \infty} (1 + p(p+1))^{\frac{1}{p}} = 1\end{aligned}$$

Since $\mathcal{D}(K) = \mathbb{D}$ and $\mathcal{D}(K)$ is always contained in the spectrum of M_z^* , we can now conclude that the spectrum of M_z is the closed unit disk. \square

We now turn our attention to the matrix form of M_ϕ . From section 5 the (n, n) th entry is ϕ_0 , the $(n+1, n)$ th entry is ϕ_1 , the $(n+2, n)$ th entry is $\phi_2 + \frac{-1}{(n+2)(n+3)}\phi_1$, and for $J \geq 3$ the $(n+J, n)$ th entry is given by $\phi_J + \frac{(-1)^{J-1}}{(n+2)(n+J+1)} \sum_{j=1}^{J-1} (-1)^{j-1} j \phi_j$ or

$$\phi_J + \frac{(-1)^{J-1}}{(n+2)(n+J+1)} [\phi_1 - 2\phi_2 + 3\phi_3 - \dots + (-1)^J (J-1)\phi_{J-1}].$$

Let P_N be the truncation of ϕ to the span of $\{1, z, \dots, z^N\}$. So $(P_N \phi)(z) = \sum_{n=0}^N \phi_n z^n$ and $(P_N \phi)'(z) = \sum_{n=1}^N n \phi_n z^{n-1}$. For $J \geq 1$ we can now express the $(n+J, n)$ th entry of M_ϕ by $\phi_J + \frac{(-1)^{J-1}}{(n+2)(n+J+1)} (P_{J-1} \phi)'(-1)$. Thus $\widehat{M}_\phi = T_\phi + S_\phi$ where T_ϕ is the usual Toeplitz matrix with symbol ϕ and S_ϕ is given by

$$S_\phi = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & \frac{-1}{2} & 0 & \ddots \\ 0 & 0 & \frac{1}{3} & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} 0 & 0 & \cdots \\ 0 & 0 & \ddots \\ (P_1 \phi)'(-1) & 0 & \ddots \\ (P_2 \phi)'(-1) & (P_1 \phi)'(-1) & \ddots \\ (P_3 \phi)'(-1) & (P_2 \phi)'(-1) & \ddots \\ \vdots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \frac{-1}{2} & 0 & 0 & \cdots \\ 0 & \frac{1}{3} & 0 & \ddots \\ 0 & 0 & \frac{-1}{4} & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

The following theorem characterizes the commutant of M_z and establishes that $\{\phi : M_\phi \text{ is in the commutant of } M_z\}$ is strictly contained in $H^\infty(\mathbb{D})$.

Theorem 8. *If $f_n(z) = (1 + \frac{n+1}{n+2}z)z^n$ for $n \geq 0$ and ϕ is an analytic symbol on $\mathbb{D} = \mathcal{D}(K)$, then*

1. $\|T_\phi\| \leq \|M_\phi\|$;
2. *The operator M_ϕ is bounded if and only if both T_ϕ and S_ϕ are bounded;*

3. $\|T_\phi\| + \|S_\phi\|$ is an equivalent norm to $\|M_\phi\|$;
4. T_ϕ is bounded if and only if ϕ is in $H^\infty(\mathbb{D})$;
5. S_ϕ is bounded if and only if $\sum_{n=1}^{\infty} \left[\frac{|(P_n\phi)'(-1)|}{n+1} \right]^2 < \infty$;
6. there exists a bounded analytic function ϕ such that T_ϕ is bounded but S_ϕ is not bounded.

Proof. Part (4) is a well known fact and is restated simply for completeness. To show $\|T_\phi\| \leq \|M_\phi\|$ we begin with a fact true for any operator.

Let $\{e_0, e_1, \dots\}$ be the standard basis for l^2 and for $0 \leq m \leq n$ let $Q_{m,n}$ be the projection onto the span of $\{e_m, e_{m+1}, \dots, e_n\}$. For any formal matrix A on l^2 , $Q_{m,n}AQ_{m,n}$ is bounded since outside of an $n - m + 1$ by $n - m + 1$ submatrix all entries are zero. Moreover,

$$\|A\| = \sup_{\vec{v} \in l^2, 0 \leq m \leq n} \frac{\|Q_{m,n}AQ_{m,n}\vec{v}\|}{\|\vec{v}\|}.$$

Now fix $\vec{v} \in l^2$ and $j \in \mathbb{Z}$ and note that if S is the unilateral shift, then for each n ,

$$\|Q_{0,j}T_\phi Q_{0,j}\vec{v}\| = \|Q_{n,n+j}T_\phi Q_{n,n+j}S^n\vec{v}\|.$$

However, $\lim_{n \rightarrow \infty} \|Q_{n,n+j}S_\phi Q_{n,n+j}S^n\vec{v}\| = 0$ since $Q_{n,n+j}S_\phi Q_{n,n+j}$ is zero outside of a $j + 1$ by $j + 1$ submatrix whose entries are converging uniformly with n to zero. Thus $\lim_{n \rightarrow \infty} \|Q_{n,n+j}M_\phi Q_{n,n+j}S^n\vec{v}\| = \lim_{n \rightarrow \infty} \|Q_{n,n+j}T_\phi Q_{n,n+j}S^n\vec{v}\|$ which implies $\|T_\phi\| \leq \|M_\phi\|$.

If M_ϕ is bounded, then T_ϕ is bounded by part (2). Hence $S_\phi = M_\phi - T_\phi$ is bounded. On the other hand if both T_ϕ and S_ϕ are bounded, then $M_\phi = T_\phi + S_\phi$ is bounded.

The fact that $\|T_\phi\| + \|S_\phi\|$ is an equivalent norm to $\|M_\phi\|$ follows from

$$\begin{aligned} (\|T_\phi\| + \|S_\phi\|)/3 &\leq (\|M_\phi\| + \|M_\phi - T_\phi\|)/3 \leq (\|M_\phi\| + \|M_\phi\| + \|T_\phi\|)/3 \\ &\leq 3\|M_\phi\|/3 \leq \|T_\phi + S_\phi\| \leq \|T_\phi\| + \|S_\phi\|. \end{aligned}$$

Assuming S_ϕ is bounded, then the l^2 norm of the first column of S_ϕ is finite. Thus $\frac{1}{2} \sum_{n=1}^{\infty} \left[\frac{|(P_n\phi)'(-1)|}{n+2} \right]^2$ converges and hence $\sum_{n=1}^{\infty} \left[\frac{|(P_n\phi)'(-1)|}{n+1} \right]^2$ converges. Conversely assume $\alpha = \sum_{n=1}^{\infty} \left[\frac{|(P_n\phi)'(-1)|}{n+1} \right]^2 < \infty$. To show S_ϕ is bounded it

suffices to show the entries of S_ϕ are square summable. To this end note that

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} \left[\frac{|(P_{j-1}\phi)'(-1)|}{(n+2)(n+j+1)} \right]^2 \leq \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} \left[\frac{|(P_{j-1}\phi)'(-1)|}{(n+1)(j)} \right]^2 \\ & \leq \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} \sum_{j=1}^{\infty} \left[\frac{|(P_{j-1}\phi)'(-1)|}{j} \right]^2 = \alpha \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} < \infty. \end{aligned}$$

For $0 < \epsilon \leq \frac{1}{2}$, let $\phi(z) = \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n^{1+\epsilon}}$ and note $\|\phi\|_{\infty} \leq \sum_{n=1}^{\infty} \frac{1}{n^{1+\epsilon}} < \infty$. A short computation shows $|(P_n\phi)'(-1)| = \sum_{k=1}^{n-1} k^{-\epsilon} \geq \int_1^{n-1} x^{-\epsilon} dx = \frac{(n-1)^{1-\epsilon} - 1}{1-\epsilon}$. Thus $\sum_{n=1}^{\infty} \left| \frac{(P_n\phi)'(-1)}{n+1} \right|^2 \geq \frac{1}{(1-\epsilon)^2} \sum_{n=1}^{\infty} \frac{((n-1)^{1-\epsilon} - 1)^2}{(n+1)^2}$. This last series is comparable to the series $\sum \frac{1}{n^{2\epsilon}}$ which diverges since $2\epsilon < 1$. Thus T_ϕ is bounded but S_ϕ is not. \square

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